Some Orbital Mechanics Exercises on the Acceleration of an Orbital Body

1. Consider the radial-transverse component form (see the Appendix) of the expression for the velocity $\mathbf{v}$ of the orbital body:

$$\mathbf{v} = \frac{v_\text{o}}{1+e}\left[e\sin\theta\,\mathbf{e}_r + (1+e\cos\theta)\,\mathbf{e}_\theta\right]$$

Based upon the definition for the acceleration $\mathbf{a}$ of the orbital body, utilize this expression for $\mathbf{v}$ to obtain the following expression for $\mathbf{a}$:

$$\mathbf{a} = -\frac{\xi}{r^2}\,\mathbf{e}_r$$

(Hint: $\dot{\theta} = h_\text{o}/r^2$, $v_\text{o} = \sqrt{\xi\,(1+e)/r_\text{o}}$, and $h_\text{o} = \sqrt{\xi\,(1+e)\,r_\text{o}}$.)

2. Explain why this expression for $\mathbf{a}$ is completely expected in the case of a single attractive central force with an inverse-square-law dependence.

(Hint: Consider *Newton's second law of motion*.)

3. Show that the expression for $\mathbf{a}$ indicated above can be recast as

$$\mathbf{a} = -\frac{v_\text{o}^2}{r_\text{o}}\,\frac{(1+e\cos\theta)^2}{(1+e)^3}\,\mathbf{e}_r$$

This latter expression can be utilized to determine $\mathbf{a}$ at any location along the orbital path (which is identified by θ) or at any instant during the orbital motion (from which θ can be evaluated by means of the final results for the solution to the *Kepler problem*).

(Hint: $r = r_\text{o}\,(1+e)/(1+e\cos\theta)$.)

Spoiler Warning: Exercise Solutions/Answers appear on back of sheet.

Exercise Solutions/Answers

1. Since the acceleration $\mathbf{a}$ is (by definition) the time-derivative of the velocity $\mathbf{v}$:

$$\begin{aligned}
\mathbf{a} &= \frac{d}{dt}\left\{\frac{v_o}{1+e}\left[e\sin\theta\,\mathbf{e}_r + (1+e\cos\theta)\,\mathbf{e}_\theta\right]\right\} \\
&= \frac{d}{d\theta}\left\{\frac{v_o}{1+e}\left[e\sin\theta\,\mathbf{e}_r + (1+e\cos\theta)\,\mathbf{e}_\theta\right]\right\}\frac{d\theta}{dt} \\
&= \frac{v_o}{1+e}\,\dot{\theta}\left[e\cos\theta\,\mathbf{e}_r + e\sin\theta\,\frac{d\mathbf{e}_r}{d\theta} - e\sin\theta\,\mathbf{e}_\theta + (1+e\cos\theta)\,\frac{d\mathbf{e}_\theta}{d\theta}\right] \\
&= \frac{v_o}{1+e}\,\dot{\theta}\left[e\cos\theta\,\mathbf{e}_r + e\sin\theta\,\mathbf{e}_\theta - e\sin\theta\,\mathbf{e}_\theta - (1+e\cos\theta)\,\mathbf{e}_r\right] \\
&= -\frac{v_o}{1+e}\,\dot{\theta}\,\mathbf{e}_r
\end{aligned}$$

But from the Appendix, $\dot{\theta} = h_o/r^2$, $v_o = \sqrt{\xi\,(1+e)/r_o}$, and $h_o = \sqrt{\xi\,(1+e)\,r_o}$. Thus,

$$\begin{aligned}
\mathbf{a} &= -\frac{v_o}{1+e}\,\frac{h_o}{r^2}\,\mathbf{e}_r \\
&= -\frac{\xi\,(1+e)}{(1+e)\,r^2}\,\mathbf{e}_r \\
&= -\frac{\xi}{r^2}\,\mathbf{e}_r
\end{aligned}$$

2. From elementary physics, $\mathbf{F} = m\,\mathbf{a}$, where $\mathbf{F}$ denotes the *total external force* that acts upon the orbital body. As a result, $\mathbf{a} = \mathbf{F}/m$. But in the case of a single attractive central force with an inverse-square-law dependence, $\mathbf{F} = -\left(m\,\xi/r^2\right)\mathbf{e}_r$, and so it must follow that $\mathbf{a} = -\left(\xi/r^2\right)\mathbf{e}_r$.

3. Since $r = r_o\,(1+e)/(1+e\cos\theta)$, and $v_o = \sqrt{\xi\,(1+e)/r_o} \;\Rightarrow\; \xi = \left(r_o\,v_o^2\right)/(1+e)$,

$$\begin{aligned}
\mathbf{a} &= -\frac{r_o\,v_o^2}{1+e}\,\frac{(1+e\cos\theta)^2}{r_o^2\,(1+e)^2}\,\mathbf{e}_r \\
&= -\frac{v_o^2}{r_o}\,\frac{(1+e\cos\theta)^2}{(1+e)^3}\,\mathbf{e}_r
\end{aligned}$$

which expresses $\mathbf{a}$ entirely as a function of θ. The factor v_o^2/r_o is a *characteristic acceleration* for the orbital body, which has the form of a *centripetal acceleration*.