Simplify: Not As Easy As It Sounds

Michael Price, University of Oregon

April 30, 2011

"The essence of mathematics is not to make simple things complicated, but to make complicated things simple."

Stanley Gudder

What We Ask

Simplify.

1. $5x - 3x + x$

What We Ask

Simplify.

1. $5x - 3x + x = 3x$

What We Ask

Simplify.

1. $5x - 3x + x = 3x$
2. $\dfrac{3x - 6}{x - 2}$

What We Ask

Simplify.

1. $5x - 3x + x = 3x$
2. $\dfrac{3x-6}{x-2} = 3$

What We Ask

Simplify.

1. $5x - 3x + x = 3x$
2. $\dfrac{3x-6}{x-2} = 3 \ , \ x \neq 2$

What We Ask

Simplify.

1. $5x - 3x + x = 3x$
2. $\dfrac{3x - 6}{x - 2} = 3 \ , \ x \neq 2$
3. $a^2 - b^2$

What We Ask

Simplify.

1. $5x - 3x + x = 3x$
2. $\dfrac{3x-6}{x-2} = 3 \ , \ x \neq 2$
3. $a^2 - b^2 = (a+b)(a-b)$

What We Ask

Simplify.

1. $5x - 3x + x = 3x$
2. $\dfrac{3x-6}{x-2} = 3 \ , \ x \neq 2$
3. $a^2 - b^2 = (a+b)(a-b)$
4. $x^2(2x-1) + 7(2x-1)$

What We Ask

Simplify.

1. $5x - 3x + x = 3x$
2. $\dfrac{3x-6}{x-2} = 3$, $x \neq 2$
3. $a^2 - b^2 = (a+b)(a-b)$
4. $x^2(2x-1) + 7(2x-1) = 2x^3 - x^2 + 14x - 7$

What We Ask

Simplify.

1. $5x - 3x + x = 3x$
2. $\dfrac{3x-6}{x-2} = 3\ ,\ x \neq 2$
3. $a^2 - b^2 = (a+b)(a-b)$
4. $x^2(2x-1) + 7(2x-1) = 2x^3 - x^2 + 14x - 7$ or $x^2(2x-1) + 7(2x-1) = (2x-1)(x^2+7)$

Why We Ask It

Arguments in favor:

- It's easy to write "Simplify" in the header as instructions for several problems.

Why We Ask It

Arguments in favor:

- It's easy to write "Simplify" in the header as instructions for several problems.
- In the science of mathematics, 'simplest' is an objective term.

Why We Ask It

Arguments in favor:

- It's easy to write "Simplify" in the header as instructions for several problems.
- In the science of mathematics, 'simplest' is an objective term.
- Giving a more detailed explanation spoils the conceptual component of students choosing an appropriate strategy.

What We Do With It

Assign scores to the following sample student work.

In each case, simplify. Each question is worth 5 points.

What We Do With It

Assign scores to the following sample student work.

In each case, simplify. Each question is worth 5 points.

1. $\sqrt{1 + \tan^2 x}$

What We Do With It

Assign scores to the following sample student work.

In each case, simplify. Each question is worth 5 points.

1. $\sqrt{1+\tan^2 x}$
   $\sqrt{1+\tan^2 x} = \sqrt{\sec^2 x} = \sec x.$

What We Do With It

Assign scores to the following sample student work.

In each case, simplify. Each question is worth 5 points.

1. $\sqrt{1+\tan^2 x}$
   $\sqrt{1+\tan^2 x} = \sqrt{\sec^2 x} = \sec x$.
   We wanted $|\sec x|$.

What We Do With It

Assign scores to the following sample student work.

In each case, simplify. Each question is worth 5 points.

1. $\sqrt{1+\tan^2 x}$
   $\sqrt{1+\tan^2 x} = \sqrt{\sec^2 x} = \sec x.$
   We wanted $|\sec x|$.
2. $|5 - 6\pi|$

What We Do With It

Assign scores to the following sample student work.

In each case, simplify. Each question is worth 5 points.

1. $\sqrt{1+\tan^2 x}$
   $\sqrt{1+\tan^2 x} = \sqrt{\sec^2 x} = \sec x$.
   We wanted $|\sec x|$.
2. $|5-6\pi|$
   $|5-6\pi| = 13.84955592$

What We Do With It

Assign scores to the following sample student work.

In each case, simplify. Each question is worth 5 points.

1. $\sqrt{1+\tan^2 x}$
   $\sqrt{1+\tan^2 x} = \sqrt{\sec^2 x} = \sec x$.
   We wanted $|\sec x|$.
2. $|5-6\pi|$
   $|5-6\pi| = 13.84955592$
   We wanted $6\pi - 5$.

What We Do With It

Assign scores to the following sample student work.

In each case, simplify. Each question is worth 5 points.

1. $\sqrt{1+\tan^2 x}$
   $\sqrt{1+\tan^2 x} = \sqrt{\sec^2 x} = \sec x$.
   We wanted $|\sec x|$.
2. $|5-6\pi|$
   $|5-6\pi| = 13.84955592$
   We wanted $6\pi - 5$.
3. $x(y+2z-3) - yx$

What We Do With It

Assign scores to the following sample student work.

In each case, simplify. Each question is worth 5 points.

1. $\sqrt{1+\tan^2 x}$
   $\sqrt{1+\tan^2 x} = \sqrt{\sec^2 x} = \sec x$.
   We wanted $|\sec x|$.
2. $|5 - 6\pi|$
   $|5 - 6\pi| = 13.84955592$
   We wanted $6\pi - 5$.
3. $x(y + 2z - 3) - yx$
   $x(y + 2z - 3) - yx = 2xz - 3x$

How We Can Adjust It

What rephrasing can be implemented in order to correct for these potential misunderstandings?

How We Can Adjust It

What rephrasing can be implemented in order to correct for these potential misunderstandings?

- Illustrate the form in which the final answer should be expressed. e.g. “Write your answer as a product of linear factors”.

How We Can Adjust It

What rephrasing can be implemented in order to correct for these potential misunderstandings?

- Illustrate the form in which the final answer should be expressed. e.g. "Write your answer as a product of linear factors".
- Give reminders about initial and final expressions being equivalent on the same set of real numbers.

How We Can Adjust It

What rephrasing can be implemented in order to correct for these potential misunderstandings?

- Illustrate the form in which the final answer should be expressed. e.g. "Write your answer as a product of linear factors".
- Give reminders about initial and final expressions being equivalent on the same set of real numbers.

Do we lose too much student contextual understanding by giving more definitive instructions?

How We Can Adjust It

What rephrasing can be implemented in order to correct for these potential misunderstandings?

- Illustrate the form in which the final answer should be expressed. e.g. "Write your answer as a product of linear factors".
- Give reminders about initial and final expressions being equivalent on the same set of real numbers.

Do we lose too much student contextual understanding by giving more definitive instructions?

- Specify that answers should be exact.

How We Can Adjust It

What rephrasing can be implemented in order to correct for these potential misunderstandings?

- Illustrate the form in which the final answer should be expressed. e.g. "Write your answer as a product of linear factors".
- Give reminders about initial and final expressions being equivalent on the same set of real numbers.

Do we lose too much student contextual understanding by giving more definitive instructions?

- Specify that answers should be exact.

Perhaps we want students to know the difference implicitly between exact and approximate answers.

How We Can Adjust It

What rephrasing can be implemented in order to correct for these potential misunderstandings?

- Illustrate the form in which the final answer should be expressed. e.g. “Write your answer as a product of linear factors”.
- Give reminders about initial and final expressions being equivalent on the same set of real numbers.

Do we lose too much student contextual understanding by giving more definitive instructions?

- Specify that answers should be exact.

Perhaps we want students to know the difference implicitly between exact and approximate answers.

- Include “Show your work”, or something to that effect, in the instructions.

How We Can Adjust It

What rephrasing can be implemented in order to correct for these potential misunderstandings?

- Illustrate the form in which the final answer should be expressed. e.g. "Write your answer as a product of linear factors".
- Give reminders about initial and final expressions being equivalent on the same set of real numbers.

Do we lose too much student contextual understanding by giving more definitive instructions?

- Specify that answers should be exact.

Perhaps we want students to know the difference implicitly between exact and approximate answers.

- Include "Show your work", or something to that effect, in the instructions.

Really? All of their work?

How We Can Adjust It

Consider the full justification for the simplification of $x(y+2z-3)-yx$:

How We Can Adjust It

Consider the full justification for the simplification of $x(y+2z-3)-yx$:

$x(y+2z-3)-yx$

$= x(y+2z+-3)+-(yx)$ Uniqueness of the additive inverse

How We Can Adjust It

Consider the full justification for the simplification of $x(y+2z-3)-yx$:

$x(y+2z-3)-yx$	
$= x(y+2z+-3)+-(yx)$	Uniqueness of the additive inverse
$= x[y+(2z+-3)]+-(yx)$	Associativity of $+$

How We Can Adjust It

Consider the full justification for the simplification of $x(y+2z-3)-yx$:

$x(y+2z-3)-yx$	
$= x(y+2z+-3)+-(yx)$	Uniqueness of the additive inverse
$= x[y+(2z+-3)]+-(yx)$	Associativity of $+$
$= xy+x(2z+-3)-(yx)$	Distributive property of $\cdot$ over $+$

How We Can Adjust It

Consider the full justification for the simplification of $x(y+2z-3)-yx$:

$x(y+2z-3)-yx$	
$= x(y+2z+-3)+-(yx)$	Uniqueness of the additive inverse
$= x[y+(2z+-3)]+-(yx)$	Associativity of $+$
$= xy+x(2z+-3)-(yx)$	Distributive property of $\cdot$ over $+$
$= xy+x\cdot(2z)+x\cdot(-3)+-(yx)$	Distributive property of $\cdot$ over $+$

How We Can Adjust It

Consider the full justification for the simplification of $x(y+2z-3)-yx$:

$x(y+2z-3)-yx$	
$= x(y+2z+-3)+-(yx)$	Uniqueness of the additive inverse
$= x[y+(2z+-3)]+-(yx)$	Associativity of $+$
$= xy+x(2z+-3)-(yx)$	Distributive property of $\cdot$ over $+$
$= xy+x\cdot(2z)+x\cdot(-3)+-(yx)$	Distributive property of $\cdot$ over $+$
$= xy+(2\cdot x)\cdot z+(-3)\cdot x+-(yx)$	Associativity and commutativity of $\cdot$

How We Can Adjust It

Consider the full justification for the simplification of $x(y+2z-3)-yx$:

$x(y+2z-3)-yx$	
$=x(y+2z+-3)+-(yx)$	Uniqueness of the additive inverse
$=x[y+(2z+-3)]+-(yx)$	Associativity of $+$
$=xy+x(2z+-3)-(yx)$	Distributive property of $\cdot$ over $+$
$=xy+x\cdot(2z)+x\cdot(-3)+-(yx)$	Distributive property of $\cdot$ over $+$
$=xy+(2\cdot x)\cdot z+(-3)\cdot x+-(yx)$	Associativity and commutativity of $\cdot$
$=xy+2xz+-3x+-(xy)$	Commutativity of $\cdot$
$=xy+-(xy)+2xz+-3x$	Associativity and commutativity of $+$
$=0+2xz+-3x$	Defn'n of additive inverse
$=2xz+-3x$	Defn'n of additive identity
$=2xz-3x$	Uniqueness of the additive inverse

“It is easy to generalize by diluting a little idea with a big terminology. It is much more difficult to prepare a refined and condensed extract from several good ingredients.”

George Polya

The Irony

We want students to understand from context what we expect of them.

The Irony

We want students to understand from context what we expect of them.
In some situations, this amounts to mind-reading.

The Irony

We want students to understand from context what we expect of them.
In some situations, this amounts to mind-reading.
e.g. "What is actually simpler?"
"Would a factored or expanded form be more useful?",
"How much am I actually expected to show?"

The Irony

We want students to understand from context what we expect of them.
In some situations, this amounts to mind-reading.
e.g. "What is actually simpler?"
"Would a factored or expanded form be more useful?",
"How much am I actually expected to show?"

These questions illustrate not (merely) a lack of understanding on the part of our students, but a lack of inspiration from us, the instructors.

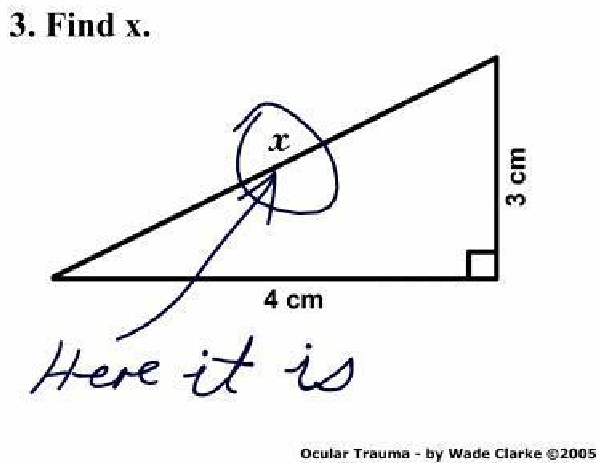
The Irony

We want students to understand from context what we expect of them.
In some situations, this amounts to mind-reading.
e.g. "What is actually simpler?"
"Would a factored or expanded form be more useful?",
"How much am I actually expected to show?"

These questions illustrate not (merely) a lack of understanding on the part of our students, but a lack of inspiration from us, the instructors.

In this case we teach them the idiosyncrasies of our particular experience in math, not the concepts that are portable.

Out of the Box Thinking



Out of the Box Thinking

The “barometer” legend.

You Know What I Meant

"On a farm there are only horses and chickens. There are 47 animals and 124 legs in total. How many of each animal are there on the farm?"

You Know What I Meant

"On a farm there are only horses and chickens. There are 47 animals and 124 legs in total. How many of each animal are there on the farm?"

Answer Key: Let h = number of horses, c = number of chickens. Then $h + c = 47$ and because horses have four legs while chickens have only two, $4h + 2c = 124$. From the first equation we have $h = 47 - c$ and by substitution $4(47 - c) + 2c = 124$.
So $188 - 2c = 124$, or

$c = \dfrac{124 - 188}{-2} = 32$. Then

$h = 47 - 32 = 15$.
Thus there are 32 chickens and 15 horses.

You Know What I Meant

"On a farm there are only horses and chickens. There are 47 animals and 124 legs in total. How many of each animal are there on the farm?"

Answer Key: Let $h =$ number of horses, $c =$ number of chickens. Then $h + c = 47$ and because horses have four legs while chickens have only two, $4h + 2c = 124$. From the first equation we have $h = 47 - c$ and by substitution $4(47 - c) + 2c = 124$.
So $188 - 2c = 124$, or

$c = \dfrac{124 - 188}{-2} = 32$. Then

$h = 47 - 32 = 15$.
Thus there are 32 chickens and 15 horses.

Student Solution: 10 horses, 37 chickens? ~~40 + 74~~.
$\boxed{\text{15 horses, 32 chickens?}}$, $60 + 64$ ✓

You Know What I Meant

What do we do with such responses?

You Know What I Meant

What do we do with such responses?

Persist? e.g. Award some partial credit for showing some understanding, but not using a technique from the course.

You Know What I Meant

What do we do with such responses?

Persist? e.g. Award some partial credit for showing some understanding, but not using a technique from the course.

Salvage? e.g. Append “You must use an algebraic process in your solution.”

You Know What I Meant

What do we do with such responses?

Persist? e.g. Award some partial credit for showing some understanding, but not using a technique from the course.

Salvage? e.g. Append "You must use an algebraic process in your solution."

Abandon?

Mary buys a lump sum value of bonds which accrue interest annually at a 3% rate. She continuously cashs out bonds such that by the end of each year she has taken out $\$80,000$. Let the value of the bonds t years after purchase be given by $V(t)$ thousand dollars.

(a) Explain why $\frac{dV}{dt} = 0.03V - 80$.

(b) Mary cashes out the full value of the bonds after exactly five years have passed. How much did she originally purchase?

Mary buys a lump sum value of bonds which accrue interest annually at a 3% rate. She continuously cashs out bonds such that by the end of each year she has taken out $\$80,000$. Let the value of the bonds t years after purchase be given by $V(t)$ thousand dollars.

(a) Explain why $\frac{dV}{dt} = 0.03V - 80$.

(b) Mary cashes out the full value of the bonds after exactly five years have passed. How much did she originally purchase?

Answer Key (b): Solve the differential equation for the particular solution $V(5) = 0$, then compute $V(0)$.

Mary buys a lump sum value of bonds which accrue interest annually at a 3% rate. She continuously cashs out bonds such that by the end of each year she has taken out $\$80,000$. Let the value of the bonds t years after purchase be given by $V(t)$ thousand dollars.

(a) Explain why $\frac{dV}{dt} = 0.03V - 80$.

(b) Mary cashes out the full value of the bonds after exactly five years have passed. How much did she originally purchase?

Answer Key (b): Solve the differential equation for the particular solution $V(5) = 0$, then compute $V(0)$.

Solution Solutions (b): About 15% used the differential equation and solved it as anticipated.

Mary buys a lump sum value of bonds which accrue interest annually at a 3% rate. She continuously cashs out bonds such that by the end of each year she has taken out \$80,000. Let the value of the bonds t years after purchase be given by $V(t)$ thousand dollars.

(a) Explain why $\frac{dV}{dt} = 0.03V - 80$.

(b) Mary cashes out the full value of the bonds after exactly five years have passed. How much did she originally purchase?

Answer Key (b): Solve the differential equation for the particular solution $V(5) = 0$, then compute $V(0)$.

Solution Solutions (b): About 15% used the differential equation and solved it as anticipated.
Another 20% used the future value of a continuous income stream.

Mary buys a lump sum value of bonds which accrue interest annually at a 3% rate. She continuously cashs out bonds such that by the end of each year she has taken out \$80,000. Let the value of the bonds t years after purchase be given by $V(t)$ thousand dollars.

(a) Explain why $\dfrac{dV}{dt} = 0.03V - 80$.

(b) Mary cashes out the full value of the bonds after exactly five years have passed. How much did she originally purchase?

Answer Key (b): Solve the differential equation for the particular solution $V(5) = 0$, then compute $V(0)$.

Solution Solutions (b): About 15% used the differential equation and solved it as anticipated.
Another 20% used the future value of a continuous income stream.
About 40% used a calculation for the present value of a continuous income stream (which, it turns out, done appropriately gives a correct answer too).

Analysis

- Why did so many answer the problem in unexpected ways?

Analysis

- Why did so many answer the problem in unexpected ways? There were five formulas written on the overhead during the exam, two of them were present and future values.

Analysis

- Why did so many answer the problem in unexpected ways? There were five formulas written on the overhead during the exam, two of them were present and future values.
- What do we learn from this exercise?

Research

“Too often, attention to language in mathematics classrooms focuses on vocabulary. Language-related instruction should move beyond simple vocabulary; it should include attention to how language is used to express mathematical ideas (functional linguistics) and the development of the mathematics register (Moschkovich, 2002; Pimm; Schleppegrell)”

Research

"Too often, attention to language in mathematics classrooms focuses on vocabulary. Language-related instruction should move beyond simple vocabulary; it should include attention to how language is used to express mathematical ideas (functional linguistics) and the development of the mathematics register (Moschkovich, 2002; Pimm; Schleppegrell)"

Building on students everyday language and *bridging* to academic language is a key strategy to develop mathematical proficiency (Echevarria, Vogt & Short, 2002)

Conclusions

My suggestions:

Conclusions

My suggestions:

- Avoid unncessary use of command phrases that contradict colloquial use.

Conclusions

My suggestions:

- Avoid unncessary use of command phrases that contradict colloquial use.

  e.g. "Write the following as a sum or difference of terms." 'difference' and 'terms' are terminology that naturally intersect with colloquial speech, but they are all important math terminology and, arguably, are irreducible definitions.

Conclusions

My suggestions:

- Avoid unncessary use of command phrases that contradict colloquial use.

  e.g. "Write the following as a sum or difference of terms." 'difference' and 'terms' are terminology that naturally intersect with colloquial speech, but they are all important math terminology and, arguably, are irreducible definitions.

  "...natural language consists of a register that is colloquial, common, familiar, and includes everyday conversational language (Chamot & O'Malley, 1994; Heath, 1983; Orr, 1987). Words such as cancel, if, and table, for example, must be relearned within the mathematics register (Pimm, 1987; Wagner, 2003)"

Conclusions

- Be accommodating of other solution strategies when the students exhibit understanding. Don't be afraid of questions with many different solutions!

Conclusions

- Be accommodating of other solution strategies when the students exhibit understanding. Don't be afraid of questions with many different solutions!

  e.g. "Define functions $f(x) = x^4 + 1$ and $g(x) = 3x^2 + x + 1$. Find the degree of the polynomial $(f \circ g)(x)$."

Conclusions

- Be accommodating of other solution strategies when the students exhibit understanding. Don't be afraid of questions with many different solutions!

  e.g. "Define functions $f(x) = x^4 + 1$ and $g(x) = 3x^2 + x + 1$. Find the degree of the polynomial $(f \circ g)(x)$."

  "The first rule of teaching is to know what you are supposed to teach. The second rule of teaching is to know a little more than what you are supposed to teach. . . . Yet it should not be forgotten that a teacher of mathematics should know some mathematics, and that a teacher wishing to impart the right attitude of mind toward problems to his students should have acquired that attitude himself." (Polya, 1945).

Conclusions

- Focus on asking questions within a context that naturally defines the students expectations of the answer.

Conclusions

- Focus on asking questions within a context that naturally defines the students expectations of the answer.

  e.g. Horses and chickens. Problems initiated in words ought to be answered in words.

Conclusions

- Focus on asking questions within a context that naturally defines the students expectations of the answer.

  e.g. Horses and chickens. Problems initiated in words ought to be answered in words.

  "Affording students opportunities to practice relevant knowledge retrieval in contexts they recognize and regard as important may rest in the concept called 'anchored instruction.' Anchored instruction is a way of transposing semantically rich, shared learning environments into the classroom (Bransford, Sherwood, Hasselbring, Kinzer, & Williams, 1990; Cognition and Technology Group, 1991)" (Bottage & Hasselbring, 1993)

Sources

Bottage, Brian; Hasselbring, Ted (1993); A comparison of two approaches for teaching complex, authentic mathematics problems to adolescents in remedial math classes. Exceptional Children, Vol 59.

Brice-Heath, Shirley (1983). Ways with words: Language, life, and work in communities and classrooms. New York: McGraw-Hill; Oxford University Press.

Bransford, John; Sherwood, R.D.; Hasselbring, Ted; Kinzer, C. K.; Williams, S. M. (1990). Anchored instruction: Why we need it and how technology can help. From D. Nix & R. Sprio (Eds), Cognition, education and multimedia. Hillsdale, NJ: Erlbaum Associates.

Chamot, A.U., O'Malley, J.M. (1994) The Calla Handbook: Implementing the Cognitive Academic Language Learning Approach. White Plains, NY: Addison Wesley Longman.

Sources

Echeverria, Jana; Vogt, MaryEllen; Short, Deborah (2002). Making content comprehesible for English learners. Allyn & Bacon.
Moschkovish, J (2002). “A situated and sociocultural perspective on bilingual mathemaics learners”, Mathematical Thinking and Learning 4: 2 - 3.

Orr, Eleanor Wilson (1987). Twice as less: Black English and the performance of black students in mathematics and science. New York: W. W. Norton.

Pimm, D. (1987). Speaking mathematically: communication in mathematics classrooms. New York: Routledge & K. Paul.

Polya, G. (1945). How to Solve It: a New Aspect of Mathematical Method, Princeton, NJ, Princeton University Press.

Schleppegrell, M (2004). The language of schooling, a functional linguistics perspective. Laurence Erlbaum Associates.