Music and Mathematics

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Euclid's Introduction to the Section of the Canon

If all things were at rest and nothing moved, there must be perfect silence in the world....For motion and percussion must precede sound; so that as the immediate cause of sound is some percussion, the immediate cause of all percussion must be motion....And whereas of vibratory impulses or motions causing a percussion on the ear...the...greater number of such impulses produce the higher sounds, whilst the slower which have fewer courses and returns, produce the lower.

Sound Waves

- Sound waves are caused by the vibratory displacement of air.
- Different sources of displacement result in different sounds.

Pitch and Frequency

- The pitch of a sound is determined by the frequency of the displacement.
- Higher frequency leads to higher pitched sounds and vice versa.

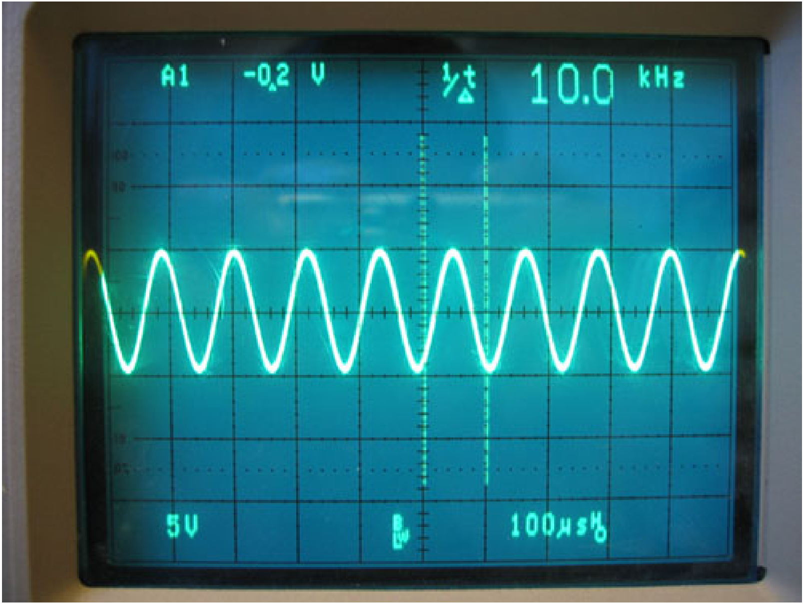
Amplitude and Intensity

- The amount of displacement in the air caused by the source of the sound is referred to as the amplitude of the sound waves.
- The greater the amplitude the louder the sound or, the greater the sound intensity.

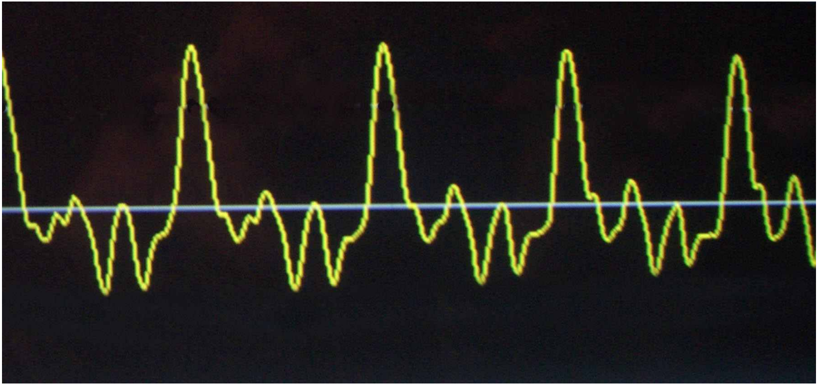
Sound Waves

- Phonodeiks, oscilloscopes and computer software can create pictures of sound waves.
- These pictures show that sound waves can be resolved into sums of perfect sine waves.

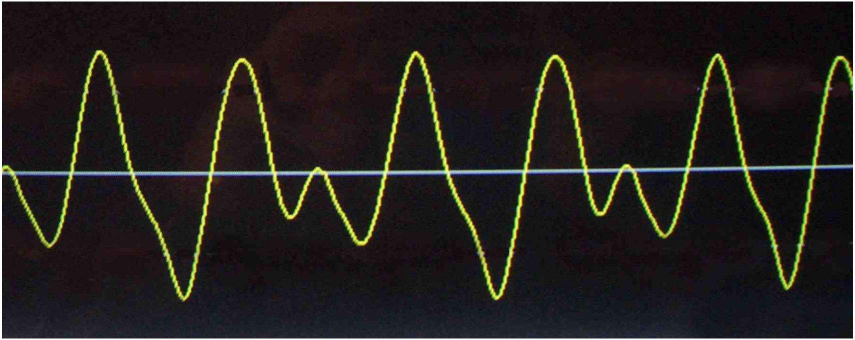
Sine Waves



Middle C - clarinet



Middle C - oboe



Pythagorean Ratios

- The ancient Pythagoreans noted that pleasing tones were produced by vibrating strings whose lengths could be represented by a ratio of whole numbers.

Pythagorean Ratios

- A vibrating string divided in two parts will produce a tone an octave higher than the original tone, and the frequency of the vibration will be double the original frequency.
- Octave frequency ratio 2:1

Pythagorean Ratios

- A vibrating string two thirds the original length will produce a tone known as a fifth in relation to the original tone, and the frequency will be 1.5 times the original.
- Fifth frequency ratio 3:2

Pythagorean Ratios

- A vibrating string three fourths the original length will produce a tone known as a fourth in relation to the original tone, and the frequency will be $4/3$ the original.
- Fourth frequency ratio 4:3

Pythagorean Ratios

- A fifth of the fourth is the octave.
- A fifth plus a fourth equals an octave.
- The frequency relationships are multiplicative so that ${}^{4}\!/_{3} * {}^{3}\!/_{2} = 2$

Pythagorean Ratios

- These tones were used in early four stringed Greek lyres, also known as the Lyre of Orpheus.
- Root, fourth, fifth, octave
- Hermes, Apollo and Orpheus were involved in the mythological development of the Greek lyre.

Greek Lyre

Pythagorean Ratios

- Pythagorean ratios can be extended to create what is known as the diatonic scale.

DO	FA	SO	DO
1:1	4:3	3:2	2:1

Pythagorean Ratios

- If we find the fifth in relation to the original fifth (SO), we find another tone, with the ratio being $^3/_2 * ^3/_2 = ^9/_4$.
- Since this is beyond the octave, we reduce the tone by an octave, cutting the frequency ratio in half and resulting in a ratio of 9:8.

Pythagorean Ratios

- Pythagorean ratios can be extended to create what is known as the diatonic scale.

DO	RE			FA	SO			DO
1:1	9:8			4:3	3:2			2:1

Pythagorean Ratios

- If we find the fifth in relation to this new tone (RE), we find another tone, with the ratio being ${}^{9}/_{8} * {}^{3}/_{2} = {}^{27}/_{16}$.

Pythagorean Ratios

- Pythagorean ratios can be extended to create what is known as the diatonic scale.

DO	RE	FA	SO	LA	DO
1:1	9:8	4:3	3:2	27:16	2:1

Pythagorean Ratios

- If we find the fifth in relation to this tone (LA), we find another tone, with the ratio being $^{27}/_{16} * ^{3}/_{2} = ^{81}/_{32}$.
- Since this is beyond the octave, we reduce the tone by an octave, cutting the frequency ratio in half and resulting in a ratio of 81:64.

Pythagorean Ratios

- Pythagorean ratios can be extended to create what is known as the diatonic scale.

DO	RE	MI	FA	SO	LA	DO
1:1	9:8	81:64	4:3	3:2	27:16	2:1

Pythagorean Ratios

- If we find the fifth in relation to this tone (MI), we find another tone, with the ratio being $^{81}/_{64} * ^{3}/_{2} = ^{243}/_{128}$.

Pythagorean Ratios

- Pythagorean ratios can be extended to create what is known as the diatonic scale.

DO	RE	MI	FA	SO	LA	TI	DO
1:1	9:8	81:64	4:3	3:2	27:16	243:128	2:1

Pythagorean Ratios

- Pythagorean ratios can be extended to create what is known as the diatonic scale.

1	1.125	1.266*	1.333	1.5	1.688*	1.898*	2
DO	RE	MI	FA	SO	LA	TI	DO
1:1	9:8	81:64	4:3	3:2	27:16	243:128	2:1

Diatonic Scales

- This is known as the Diatonic Scale.

Diatonic Scales

- The Greeks created seven diatonic scales or "modes" beginning with each of the seven tones in the Pythagorean diatonic scale.
- These modes were later adapted by medieval church musicians and became known as the church modes.

Diatonic Scales

- Notice that the divisions or ratios between the tones of the Pythagorean diatonic scale are not constant.

DO	RE	MI	FA	SO	LA	TI	DO
1:1	9:8	81:64	4:3	3:2	27:16	243:128	2:1

$$\frac{RE}{DO} = \frac{9}{8} \qquad \frac{MI}{RE} = \frac{9}{8} \qquad \frac{FA}{MI} = \frac{256}{243}$$

$$\frac{SO}{FA} = \frac{9}{8} \qquad \frac{LA}{SO} = \frac{9}{8} \qquad \frac{TI}{LA} = \frac{9}{8}$$

$$\frac{DO}{TI} = \frac{256}{243}$$

Diatonic Scales

- If we continue to create perfect fifths for each tone of the diatonic scale, we will encounter an interesting phenomenon.

$$\mathrm{DO}^{*}\ {}^{3}\!/_{2} = \mathrm{SO} \qquad \mathrm{RE}^{*}\ {}^{3}\!/_{2} = \mathrm{LA}$$

$$\mathrm{MI}^{*}\ {}^{3}\!/_{2} = \mathrm{TI} \qquad \mathrm{FA}^{*}\ {}^{3}\!/_{2} = \mathrm{DO}\ \text{(octave)}$$

$$\mathrm{SO}^{*}\ {}^{3}\!/_{2} = \mathrm{RE}\ \text{(octave)} \qquad \mathrm{LA}^{*}\ {}^{3}\!/_{2} = \mathrm{MI}\ \text{(octave)}$$

$$\mathrm{TI}^{*}\ {}^{3}\!/_{2} = ? \qquad ? = {}^{729}\!/_{256}$$

$${}^{729}\!/_{512} \approx 1.424$$

Pythagorean Ratios

- Pythagorean ratios can be extended to create what is known as the diatonic scale.

1	1.125	1.266*	1.333	1.5	1.688*	1.898*	2
DO	RE	MI	FA	SO	LA	TI	DO
1:1	9:8	81:64	4:3	3:2	27:16	243:128	2:1

1:1

4:3 3:2

C

F G

16:9 9:8

B♭ D

32:27 E♭ A 27:16

A♭ E

128:81

~

G♯ 81:64

6561:4096

C♯ B

F♯

2187:2048 243:128

729:512

Diatonic Scale

- In the previous diagram the Pythagorean comma appears as the difference between the notes A♭ and G♯.

$$\frac{{}^{6561}/_{4096}}{{}^{128}/_{81}} = \frac{3^{12}}{2^{19}} = \frac{531441}{524288}$$

Diatonic Scale

- This Pythagorean comma also appears as the difference between the major tone ratio of 9:8 and the combination of two semitone ratios of 256:243

$$\left(\frac{256}{243}\right)^2 = \frac{2^{16}}{3^{10}}$$

$$\frac{{}^{9}\!/_{8}}{{}^{2^{16}}\!/_{3^{10}}} = \frac{3^{12}}{2^{19}} = \frac{531441}{524288}$$

DO	RE	MI	FA	SO	LA	TI	DO
1:1	9:8	81:64	4:3	3:2	27:16	243:128	2:1

$$\frac{RE}{DO} = \frac{9}{8} \qquad \frac{MI}{RE} = \frac{9}{8} \qquad \frac{FA}{MI} = \frac{256}{243}$$

$$\frac{SO}{FA} = \frac{9}{8} \qquad \frac{LA}{SO} = \frac{9}{8} \qquad \frac{TI}{LA} = \frac{9}{8}$$

$$\frac{DO}{TI} = \frac{256}{243}$$

Historical Use of Scales

- The Pythagorean or Diatonic scales were used for many years.
- European musicians and composers made various adjustments to the diatonic scales including meantone temperament and well temperament.

Equal Temperament

- An equal tempered scale would erase the difference between G$\sharp$ and A$\flat$ that arises in the just intonation of the Diatonic scale. An equal temperament dividing the octave into 12 tones gives a good approximation of the diatonic scale.

Just Intonation

- Just intonation produces what are known as "wolf intervals." These are frequencies which should sound "nice" together, but don't because of the Pythagorean comma.

Just Intonation

- Equal temperament eases these wolf intervals, however, equal temperament shifts the frequencies off of the perfect 4ths and perfect 5ths.

Equal Temperament

- A Chinese work from about 400 AD shows string lengths in the required ratios to give a very good approximation of 12 tone equal temperament.
- Ho-Tchheng Thyen devised a series of string lengths as follows:

Pythagorean Ratios

Root Octave

5th century Chinese approximation

900	849	802	758	715	677	638	601	570	536	509.5	479	450
1	1.06	1.12	1.19	1.26	1.33	1.41	1.497	1.58	1.68	1.766	1.88	2

Modern 12 tone equal temperament

1	1.06	1.12	1.19	1.26	1.33	1.41	1.498	1.59	1.68	1.78	1.89	2

Equal Temperament

$\sqrt[12]{2}$	Ratio	$\sqrt[12]{2}$	Ratio
1	1		
1.059	1.060	1.4983	1.4975
1.122	1.122	1.5874	1.5789
1.1892	1.1873	1.6818	1.6791
1.2599	1.2587	1.7818	1.766
1.3348	1.3294	1.8877	1.8789
1.4142	1.41066	2	2

Equal Temperament

- In the late 16th century, Zhu Zaiyu (1584) determined a method for 12 tone equal temperament based on the twelfth roots of 2 ($\sqrt[12]{2}$).

Equal Temperament

- A jesuit named Matteo Ricci was visiting China at the time. He recorded this information and returned with it to Europe, where Simon Stevin (1585) and Marin Mersenne (1636) would begin build a case for equal temperament in European music.

Equal Temperament

- In equal temperament the ratios of the twelve tones must be equal and the product of these twelve ratios must equal 2.
- $x^{12} = 2$

Equal Temperament

- In addition, the perfect fourth and perfect fifth should be well approximated within the scale.
- $x^a = 4/3$

  AND
- $x^b = 1.5$

Equal Tempered Scale

- Essentially, we are looking for a series of numbers that satisfy the following equations:
- $x^{12} = 2$
- $x^{a} \approx {}^{4}\!/_{3}$
- $x^{b} \approx 1.5$

Equal Tempered Scale

- If $x^{12} = 2$, then $x = \sqrt[12]{2} \approx 1.06$ then
- $x^5 = 2^{5/12} \approx 1.3348$ and
- $x^7 = 2^{7/12} \approx 1.4983$
- These are "pretty good" approximations of the perfect fourth and perfect fifth.

Pythagorean Ratios

Root Octave

Diatonic Scale

1	1.125	1.266*	1.333	1.5	1.688*	1.898*	2
DO	RE	MI	FA	SO	LA	TI	DO

12 tone equal temperament

1	1.06	1.12	1.19	1.26	1.33	1.41	1.498	1.59	1.68	1.78	1.89	2

Equal Tempered Scale

C	261.63 hz	F♯	370 hz
C♯	277.18 hz	G	392 hz
D	293.66 hz	G♯	415.3 hz
D♯	311.13 hz	A	440 hz
E	329.63 hz	B♭	466.16 hz
F	349.23 hz	B	493.88 hz

C 523.25 hz

1	1.125	1.266*	1.333	1.5	1.688*	1.898*	2
DO	RE	MI	FA	SO	LA	TI	DO
1:1	9:8	81:64	4:3	3:2	27:16	243:128	2:1

C	D	E	F	G	A	B	C
1	1.122	1.26	1.3348	1.4983	1.6818	1.8877	2

C ♯	D♯	F♯	G♯	B♭
1.059	1.1892	1.4142	1.5874	1.7818

Indian Scale

- Some Indian music uses a scale that has similar ratios to the diatonic scale. An Indian svara consisting of the notes

  SA RE GA MA PA DHA NI SA

  corresponds very closely to the ratios from the diatonic scale.

Diatonic Scale

DO	RE	MI	FA	SO	LA	TI	DO
1:1	9:8	81:64	4:3	3:2	27:16	243:128	2:1

Indian Scale

SA	RE	GA	MA	PA	DHA	NI	SA
1:1	9:8	5:4	4:3	3:2	5:3	15:8	2:1

Indian Scale

- However the root tone is often a different frequency than the standard western tuning of A440.

Diatonic Scale

C	D	E	F	G	A	B	C
261.6	293.7	329.6	349.2	392	440	493.9	523.2

Indian Scale

SA	RE	GA	MA	PA	DHA	NI	SA
240	270	300	320	360	400	450	480

Equal Tempered Scale

- If we return to the concept of an equal tempered scale, there are other ways to divide the octave into equal steps.

Equal Tempered Scale

- The problem can be framed by attempting to find a satisfactory solution to the following problem:

$$2^{p/q} \approx {}^{3}\!/_{2} = 1.5$$

- This represents an attempt to use an irrational number ($(\sqrt[q]{2})^p$) to approximate a rational number (${}^{3}\!/_{2}$)

Equal Tempered Scale

$$2^{p/q} \approx {}^3/_2$$

$$({}^p/_q) \log 2 \approx \log {}^3/_2$$

$$({}^p/_q) \log 2 \approx \log 3 - \log 2$$

$$({}^p/_q) \approx \frac{\log 3}{\log 2} - 1$$

Equal Tempered Scale

$$({}^{p}/{}_{q}) \approx \frac{\log 3}{\log 2} - 1 \approx 0.5849625007\ldots$$

- Other equal-tempered scales can be developed through attempts to approximate this number.

Equal Tempered Scale

- In the 19 tone scale, the 11^{th} tone is the musical fifth (1.5) and the 8^{th} tone is the musical fourth ($^4/_3$)

$$2^{11/19} \approx 1.49376$$

$$2^{8/19} \approx 1.3389$$

Equal Tempered Scale

- In the 31 tone scale, the 18^{th} tone is the musical fifth (1.5) and the 13^{th} tone is the musical fourth ($^4/_3$)

$$2^{18/31} \approx 1.4955$$

$$2^{13/31} \approx 1.33733$$

Equal Tempered Scale

- In the 41 tone scale, the 24^{th} tone is the musical fifth (1.5) and the 17^{th} tone is the musical fourth ($^4/_3$)

$$2^{24/41} \approx 1.50042$$

$$2^{17/41} \approx 1.33296$$

Equal Tempered Scale

- It is also possible to approximate the value of

$$\frac{\log 3}{\log 2} - 1 \approx 0.5849625007\ldots$$

using a process of continued fractions.

Equal Tempered Scale

- The approximation is

$$\frac{\log 3}{\log 2} - 1 = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{\ddots}}}}}}$$

Equal Tempered Scale

- This fraction can be created through successive approximation. We know the decimal approximation as 0.5849625007..., so we can see that if we approximate this value with a fraction whose numerator is 1, then the denominator must be between 1 and 2.

Equal Tempered Scale

- So,

$$\frac{\log 3}{\log 2} - 1 = \cfrac{1}{1 + \cfrac{1}{?}}$$

Each choice of a denominator leads to a successive fractional value, which continues the fraction - indefinitely in the case of an irrational number.

Equal Tempered Scale

- We can stop the process along the way for a rational approximation of the irrational value we're searching for.

Equal Tempered Scale

- The approximation is

$$\frac{\log 3}{\log 2} - 1 = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{\ddots}}}}}}$$

Equal Tempered Scale

$$\frac{\log 3}{\log 2} - 1 \approx \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + {}^{1}\!/_{2}}}} =$$

$$\cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{{}^{5}\!/_{2}}}} = \cfrac{1}{1 + \cfrac{1}{1 + {}^{2}\!/_{5}}} = \cfrac{1}{1 + \cfrac{1}{{}^{7}\!/_{5}}}$$

Equal Tempered Scale

$$\frac{\log 3}{\log 2} - 1 \approx \frac{1}{1 + \frac{1}{7/5}} = \frac{1}{1 + 5/7} = \frac{1}{12/7}$$

$$= \frac{7}{12}$$

Equal Tempered Scale

$$\frac{\log 3}{\log 2} - 1 \approx \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{3}}}}}$$

Equal Tempered Scale

$$\frac{\log 3}{\log 2} - 1 \approx \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + {}^{3}\!/_{7}}}}$$

Equal Tempered Scale

$$= \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{17/7}}} = \cfrac{1}{1+\cfrac{1}{1+7/17}} = \cfrac{1}{1+\cfrac{1}{24/17}}$$

Equal Tempered Scale

$$\frac{\log 3}{\log 2} - 1 \approx \frac{1}{1 + {}^{17}\!/_{24}} = \frac{1}{{}^{41}\!/_{24}} = {}^{24}\!/_{41}$$

Musical Scales

- This is part of the reason why the 12-tone and 41-tone scales give such good approximations of the Pythagorean tones or Just Intonation.

Pythagorean Ratios

- The Pythagoreans knew that the tones produced by vibrating strings were related to the length of the string.
- They also knew that strings in lengths of small whole number ratios produced pleasing tones when played together - musical fourth, fifth and octave.

Frequency

- The concept of vibrational frequency was considered to be related to the length of the string.
- Galileo focused on the concept of vibrational frequency as opposed to the ratio of lengths in determining musical pitch.

Frequency

"I say that the length of the strings is not the direct and immediate reason behind the forms of musical intervals, nor is their tension, nor their thickness, but rather, the ratio of the numbers of vibrations and impact of air waves that go to strike our eardrum, which likewise vibrates according to the same measure of times."

- Galileo (1638)

Frequency

- How do we determine the frequency of a vibrating string?
- By the early 1600′s Mersenne knew that the frequency and pitch of a vibrating string were related to:

Frequency

- $\ell =$ length
- $F =$ tension
- $\sigma =$ cross sectional area
- $\rho =$ density

Frequency

Building on Mersenne's work and using a pendulum analogy, Joseph Saveur and Christiaan Huygens determined the freuqency to be:

$$\nu \approx \frac{1}{2\ell}\sqrt{\frac{F}{\rho\sigma}}$$

Frequency

- Sauveur used "beats" and ratios to determine absolute frequency.
- Beats are a musical phenomenon well-known to musicians and used often in tuning instruments.

Frequency

- One definition of beats is that they are "periodic fluctuations of loudness produced by the superposition of tones of close, but not identical frequencies."

Frequency

- The number of beats per second is actually equal to the difference in the absolute frequencies of the two tones.
- There are two ways to show that this is true. One uses algebra, the other trigonometry.

Beats Per Second

- To show that the number of beats per second is equal to the difference between the frequencies of the tones, we must consider what is happening acoustically.
- One of the frequencies is higher that the other.

Beats Per Second

- One of the frequencies is higher thant the other. That means that the higher frequency will have more wavelengths per second than the lower frequency. The beats are a result of the two wavelengths coinciding to produce a momentarily louder tone.

Beats Per Second

For instance, if two tones are 8 hertz (8 wavelengths per second) and 6 hertz, then the higher frequency wavelength will have wave peaks at:

$$t = 0, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8} \text{ and } t = 1 \text{ second}$$

Beats Per Second

The lower frequency will have wave peaks at:

$$t = 0, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6} \text{ and } t = 1 \text{ second}$$

So, they will coincide at $t = 0$, $t = \frac{1}{2}$ and then $t = 1$, $t = 1.5$ and so on, or twice each second.

Beats Per Second

- In the example, when the 8 hertz wavelength had completed 4 waves, the 6 hertz wavelength had only completed 3 waves.
- When the higher frequency completes one more wave than the lower frquency, they will coincide and produce a beat.

Beats Per Second

- Any mathematician knows that one example doesn't prove anything, so let's consider the idea in general.
- What is happening is that the higher frequency "laps" the lower frequency.

Beats Per Second

So, given two frequencies f_1 and f_2, with $f_2 > f_1$, we want to find out how many wavelengths it will take for f_2 to complete one more wavelength than f_1.

Beats Per Second

So we set:

$$\frac{N}{f_1} = \frac{N+1}{f_2}$$

$$N * f_2 = (N+1) * f_1$$

Beats Per Second

Or:

$$N * f_2 = (N + 1) * f_1$$

$$N * f_2 = N * f_1 + f_1$$

$$N * f_2 - N * f_1 = f_1$$

$$N(f_2 - f_1) = f_1$$

$$N = \frac{f_1}{(f_2 - f_1)}$$

Beats Per Second

In our example f_1 was 6 and f_2 was 8, so this value for N would come out to:

$$N = \frac{6}{8-6} = \frac{6}{2} = 3$$

Beats Per Second

This is what we saw, the wavelengths coincided on the 3rd wave of the 6 hertz frequency and the 4th wave of the 8 hertz frequency.

Beats Per Second

The time period for the first beat was $\frac{3}{6}$ seconds or $\dfrac{N}{f_1}$.

$$\text{If } N = \frac{f_1}{f_2 - f_1}, \text{ then } \frac{N}{f_1} = \frac{\frac{f_1}{f_2 - f_1}}{f_1} =$$

$$\frac{f_1}{f_2 - f_1} * \frac{1}{f_1} = \frac{1}{f_2 - f_1}$$

Beats Per Second

In our example, the beats occured every $\frac{1}{2}$ second, so there were 2 beats per second. In general, the beats occur every $\dfrac{1}{f_2 - f_1}$ seconds, so there are $f_2 - f_1$ beats per second.

Beats Per Second

Showing this relationship using trigonometry uses the Sum-to-Product Identity:

$$\sin a + \sin b = 2 \sin \frac{a+b}{2} \cos \frac{a-b}{2}$$

Beats Per Second

This comes from the standard sine sum and difference identities:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

and

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

Beats Per Second

If we add these two together, we get:

$$\sin(x+y)+\sin(x-y)=2\sin x\cos y$$

Let $x+y=a$ and $x-y=b$, then

$x=\frac{a+b}{2}$ and $y=\frac{a-b}{2}$

Beats Per Second

Then:

$$\sin(x + y) + \sin(x - y) = 2 \sin x \cos y$$

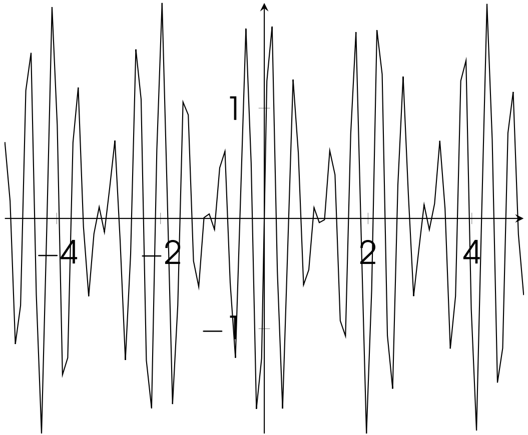
becomes

$$\sin a + \sin b = 2 \sin \frac{a+b}{2} \cos \frac{a-b}{2}$$

Beats Per Second

In the $\cos\frac{a-b}{2}$ term, we see that the addition of two sound waves ends up being identical to a sound wave with a frequency equal to the difference of the waves and a variable amplitude. The frequency is actually half the difference, but the beats occur on both maximum and minimum values, which is double the frequency of the actual wave.

Beats Per Second



Frequency

- Sauveur used "beats" and ratios to determine absolute frequency.
- We've seen that beats can determine the difference between two frequencies.

Frequency

- If we use the approximation for frequency developed by Sauveur and Huygens, we can see that the ratio of two frequencies is the inverse ratio of the their lengths, assuming that they have equal tension , cross-section, and density.

Frequency

$$\frac{f_2}{f_1} = \frac{\frac{1}{2\ell_2}\sqrt{\frac{F}{\rho\sigma}}}{\frac{1}{2\ell_1}\sqrt{\frac{F}{\rho\sigma}}}$$

$$\frac{f_2}{f_1} = \frac{\frac{1}{2\ell_2}}{\frac{1}{2\ell_1}}$$

Frequency

$$\frac{f_2}{f_1} = \frac{1}{2\ell_2} * \frac{2\ell_1}{1} = \frac{\ell_1}{\ell_2}$$

Calculating Frequency

- So, if we know the ratio of two frequencies and we know the difference of two frequencies, then we can calculate what the frequencies themselves are.
- This is one method of hand calculating the frequencies of the notes in the Eurolean equal tempered scale.

Calculating Frequency

If we have two frequencies f_1 and f_2, with $f_2 > f_1$, then:

$$f_2 - f_1 = d$$

and

$$\frac{f_2}{f_1} = r$$

Then we can calculate f_1 and f_2.

Calculating Frequency

$$\frac{f_2}{f_1} = r$$

$$\frac{f_2}{r} = f_1$$

Calculating Frequency

Then isolate f_2 from the equation

$$f_2 - f_1 = d$$

So

$$f_2 = d + f_1$$

And substitute into the other equation, giving us:

$$\frac{d + f_1}{r} = f_1$$

Calculating Frequency

$$\frac{d + f_1}{r} = f_1$$

$$d + f_1 = r * f_1$$

$$d = r * f_1 - f_1$$

$$\frac{d}{r - 1} = f_1$$

Calculating Frequency

An example -

What if we had two strings - one that was 111 cm long and one that was 110 cm long? We know that the ratio of their frequencies is the reciprocal of the ratio of their length.

Calculating Frequency

$$\frac{f_2}{f_1} = \frac{\ell_1}{\ell_2}$$

So,

$$\frac{f_2}{f_1} = \frac{111}{110}$$

Imagine that we plucked each string under equal tension and counted 2 beats per second in the sound.

Calculating Frequency

In this example

$$r = \frac{f_2}{f_1} = \frac{111}{110}$$

and

$$d = f_2 - f_1 = 2$$

Calculating Frequency

Remember that

$$\frac{d}{r-1} = f_1$$

so

$$\frac{2}{\frac{111}{110} - 1} = f_1$$

$$\frac{2}{\frac{1}{110}} = 2 * 110 = 220 = f_1$$

Calculating Frequency

This is the A note below middle C. The higher frequency sound is 222 hertz, which is somewhere between A and A$\sharp$.

Musical Scales

- As is true of many things, musical scales are dependent upon human choices and these choices can be investigated and evaluated using mathematics.

Musical Scales

- However, it is nice to just sit back and enjoy the music!