Euler Could Add

Historical Treatment of Infinite Series

ORMATYC 2019

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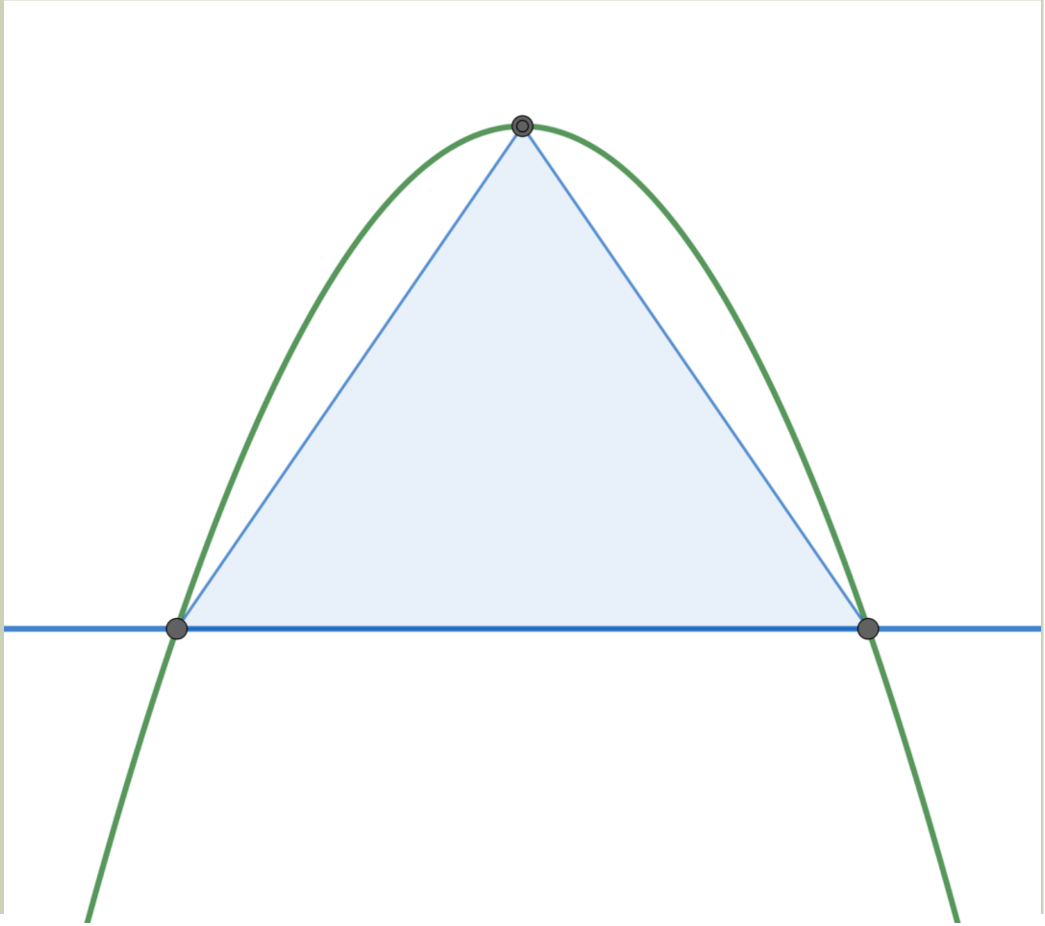
An Early Record of an Infinite Sum

Archimedes (288 BCE - 212 BCE)

Calculating the Area Under a Parabola with the Method of Exhaustion.

1. Inscribe a triangle.

   Say this triangle has area *T*.



An Early Record of an Infinite Sum

Archimedes (288 BCE - 212 BCE)

Calculating the Area Under a Parabola with the Method of Exhaustion.

1. Inscribe a triangle.

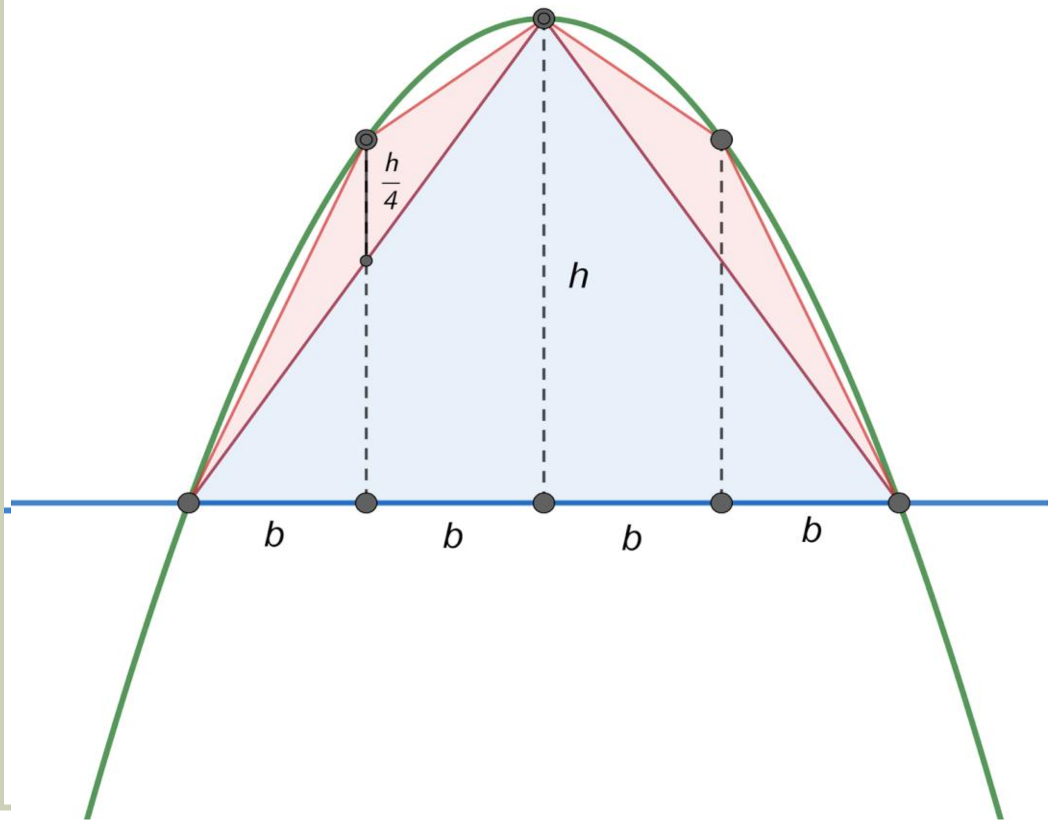
   Say this triangle has area *T*.

2. Inscribe 2 additional triangles in remaining parabolic segments.

   Each has area $\frac{T}{8}$.

   The three triangles together have an area of

   $$T + 2\frac{T}{8} = T + \frac{1}{4}T$$



An Early Record of an Infinite Sum

Archimedes (288 BCE - 212 BCE)

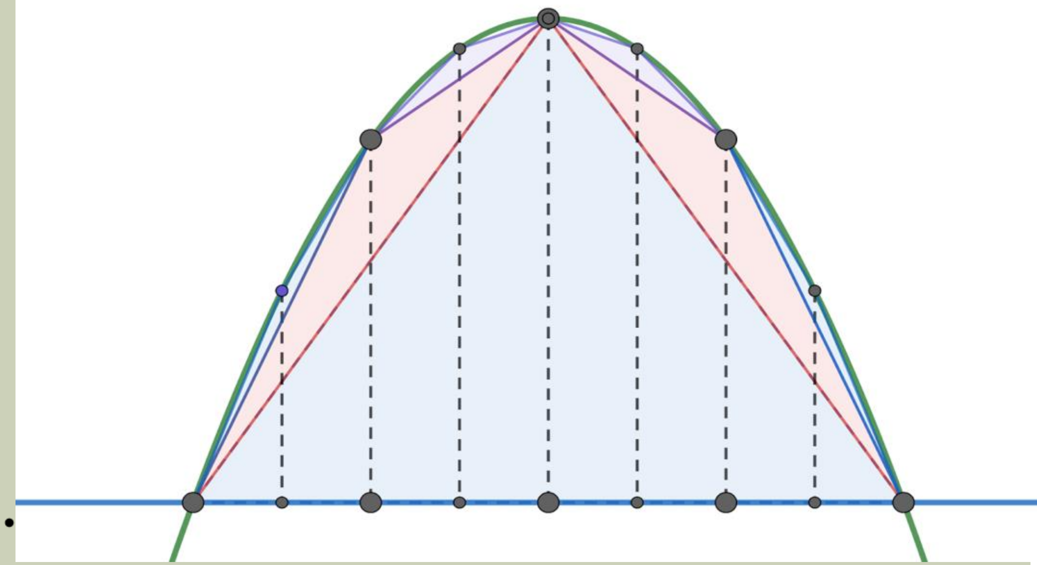
Calculating the Area Under a Parabola with the Method of Exhaustion.

Archimedes continued inscribing triangles in the remaining parabolic segments.

- Each iteration involves twice as many triangles as the last.
- The triangles are always 1/8 the area of the last.

Archimedes determined that the area of the entire parabolic segment is

$$T + 2\frac{T}{8} + 4\frac{T}{64} + 8\frac{T}{512} + \dots$$



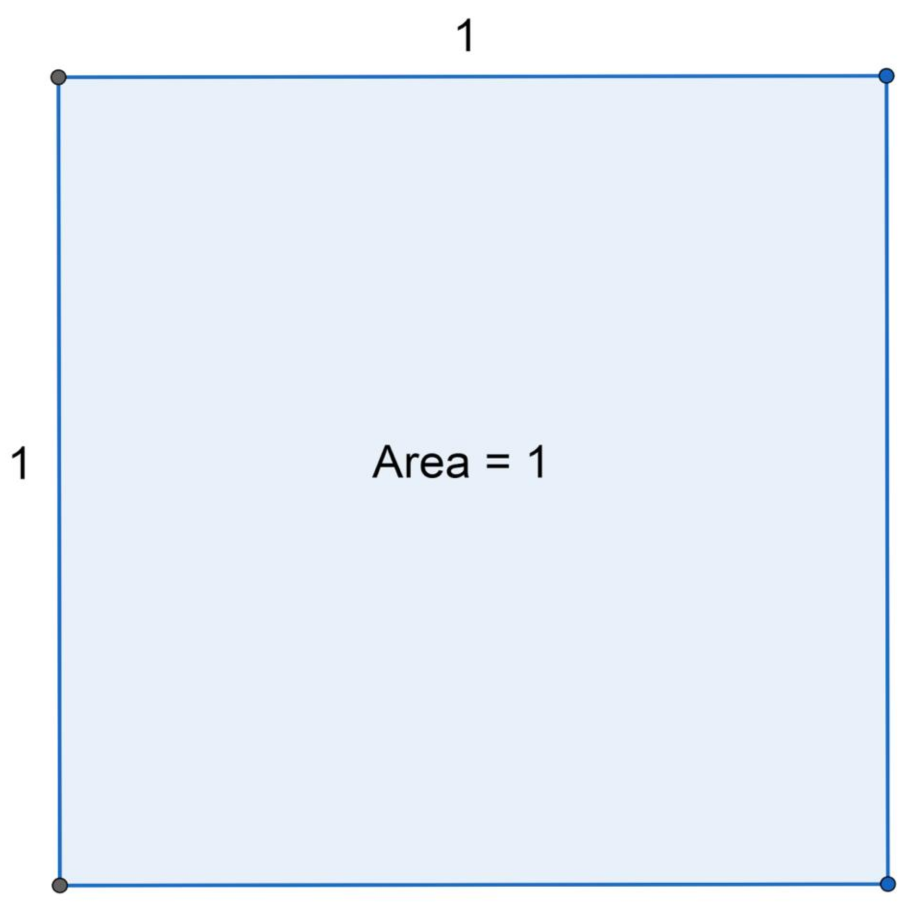
$$= T + \tfrac{1}{4}T + \tfrac{1}{16}T + \tfrac{1}{64}T + \dots$$

$$= T\left(1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots\right)$$

An Early Record of an Infinite Sum

Archimedes (288 BCE - 212 BCE)

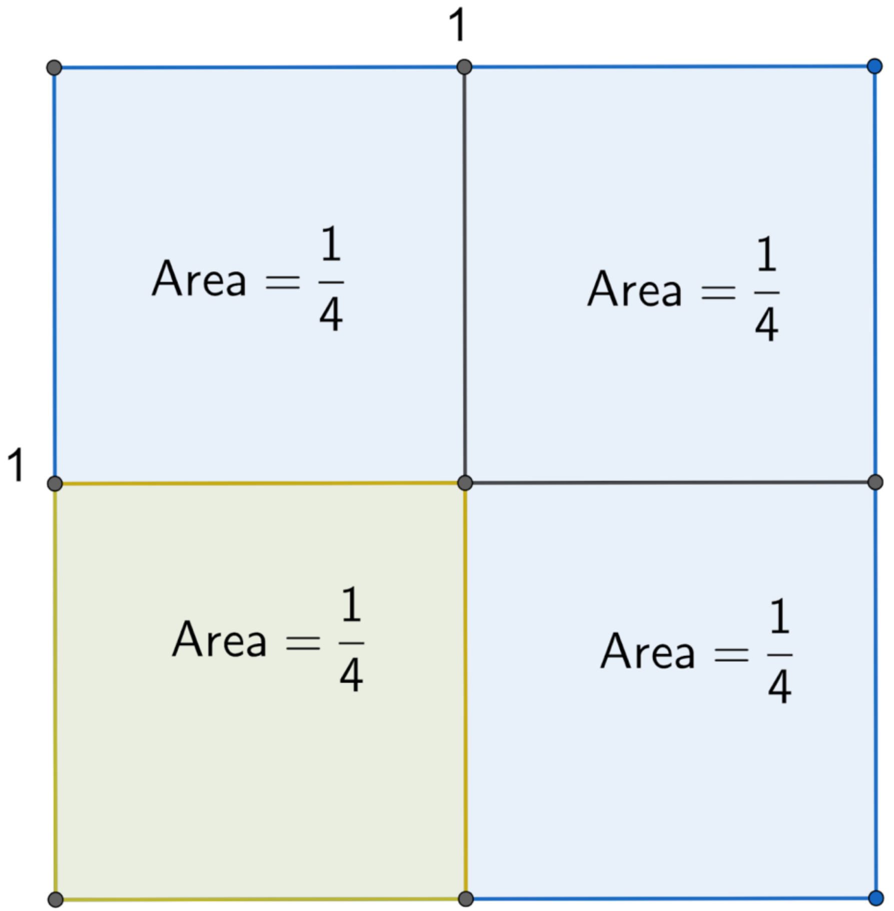
$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots$$



An Early Record of an Infinite Sum

Archimedes (288 BCE - 212 BCE)

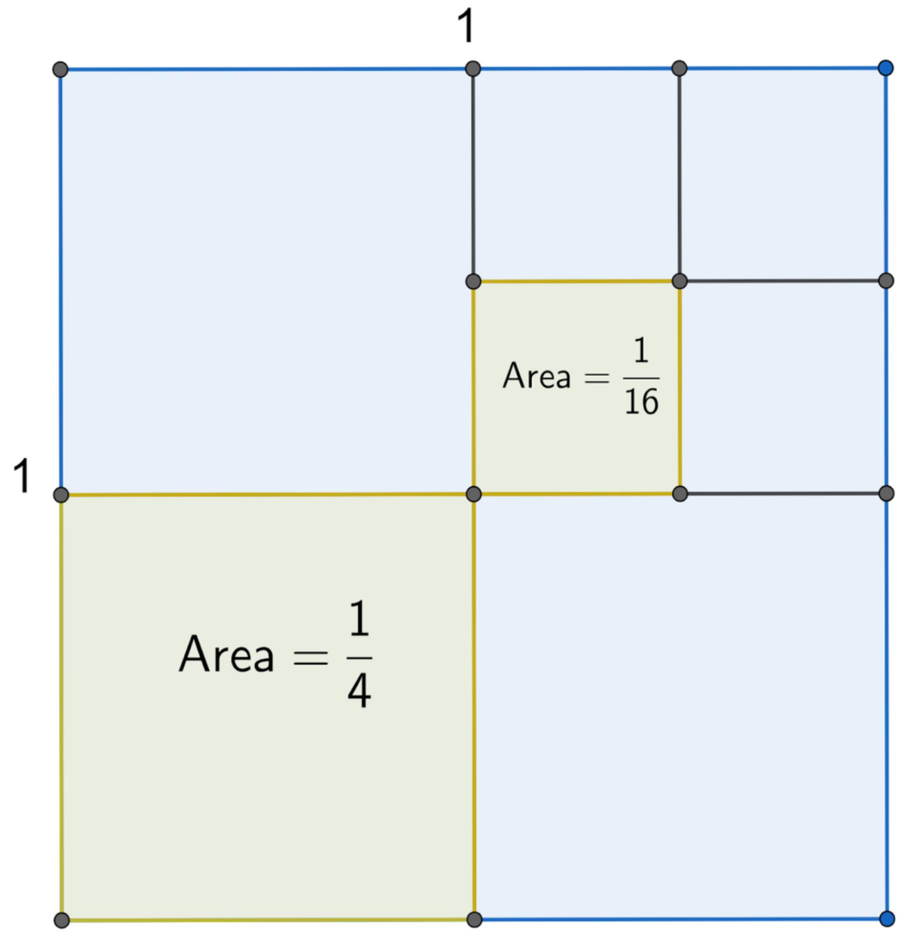
$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots$$



An Early Record of an Infinite Sum

Archimedes (288 BCE - 212 BCE)

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots$$

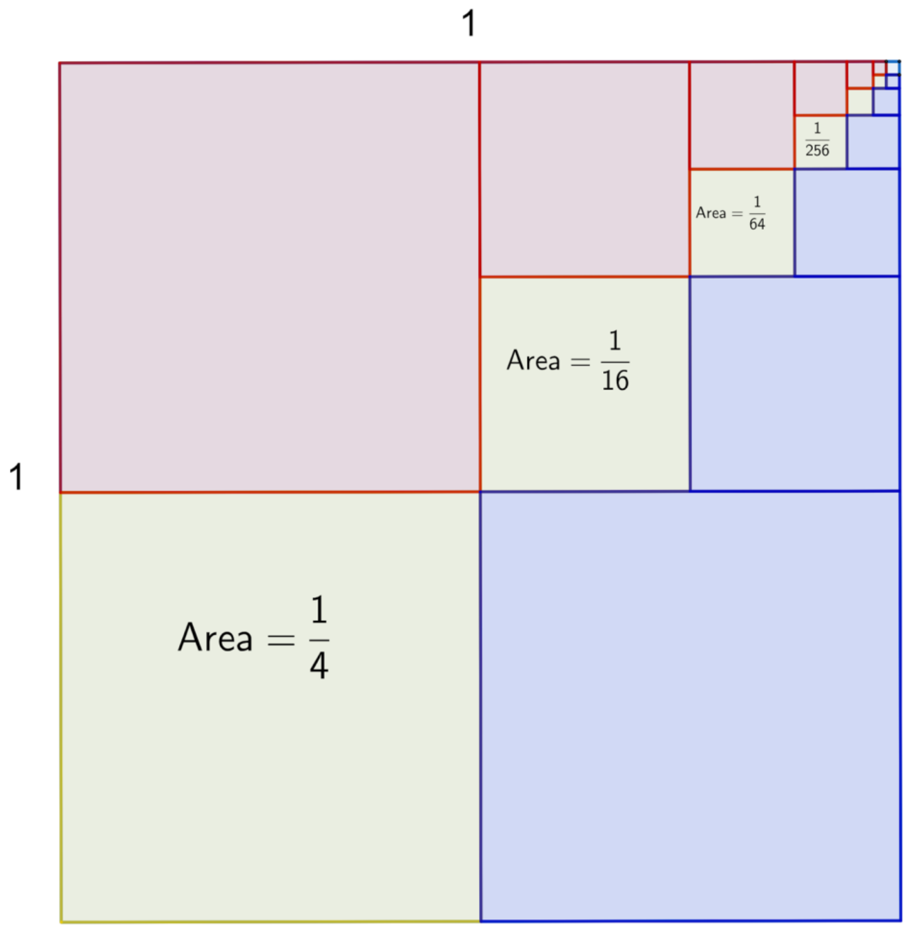


An Early Record of an Infinite Sum

Archimedes (288 BCE - 212 BCE)

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots$$

$$= \frac{1}{3}$$



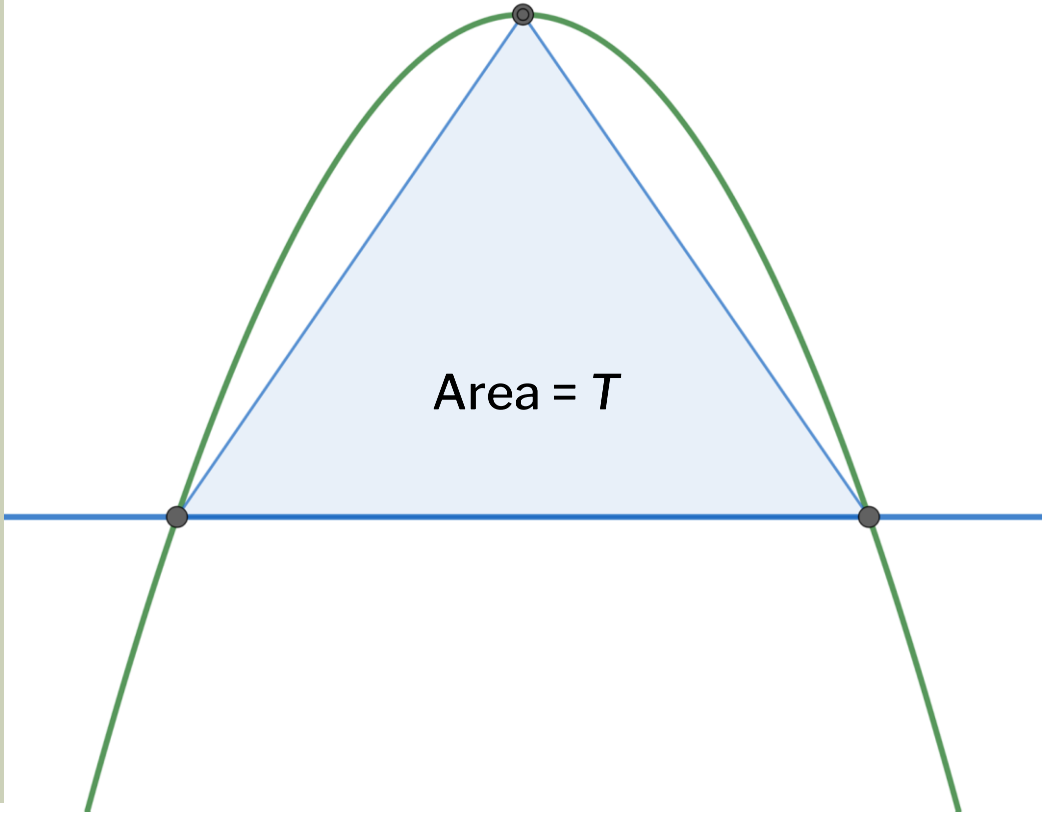
An Early Record of an Infinite Sum

Archimedes (288 BCE - 212 BCE)

Calculating the Area Under a Parabola with the Method of Exhaustion.

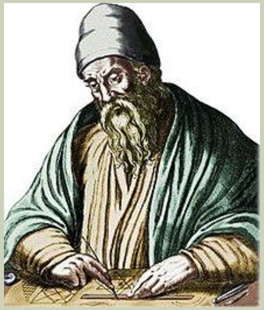
Exact area is

$$T\left(1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots\right)$$
$$=T\left(1+\frac{1}{3}\right)$$
$$=\frac{4}{3}T$$



Geometric Series

Francois Viète (1540 - 1603)



Euclid's Proposition 12, Book 5 (450 – 350 BCE):

If any number of magnitudes are proportional, then one of the antecedents is to one of the consequents as the sum of the antecedents is to the sum of the consequents.

Translation: If $a_n = r \cdot a_{n-1}$ (i.e. we have a geometric sequence), then

$$\frac{a_1}{a_2} = \frac{S_n - a_n}{S_n - a_1} = \frac{a_1 + a_2 + a_3 + \cdots + a_{n-1}}{a_2 + a_3 + \cdots + a_{n-1} + a_n}$$

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \ldots a_n$$

Geometric Series

Francois Viète (1540 - 1603)

Note that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a-b}{a} = \frac{c-d}{c}$ (assuming nonzero values)

From Euclid: If $a_n = r \cdot a_{n-1}$, then $\frac{a_1}{a_2} = \frac{S_n - a_n}{S_n - a_1}$.

This gives us $\frac{a_1 - a_2}{a_1} = \frac{S_n - a_n - (S_n - a_1)}{S_n - a_n} = \frac{a_1 - a_n}{S_n - a_n}$.

In the cases in which the terms are decreasing in magnitude ($a_n \to 0$) Viète concluded that

$$\frac{a_1 - a_2}{a_1} = \frac{a_1}{S} \quad \text{where} \quad S = \sum_{k=1}^{\infty} a_k .$$

Geometric Series

Francois Viète (1540 - 1603)

If $a_n = r \cdot a_{n-1}$ and $a_n \to 0$, then $\dfrac{a_1 - a_2}{a_1} = \dfrac{a_1}{S}$.

Or, more usefully, $\dfrac{S}{a_1} = \dfrac{a_1}{a_1 - a_2}$, which implies $S = \dfrac{a_1^2}{a_1 - a_2}$.

Archimedes series: $1 + \frac{1}{4} + \frac{1}{16} + \ldots$

$$1 + \frac{1}{4} + \frac{1}{16} + \ldots = \frac{1^2}{1 - \frac{1}{4}} = \frac{1}{3/4} = \frac{4}{3}$$

Geometric Series

Francois Viète (1540 - 1603)

Viète was very close to the modern formula commonly used for Geometric series.

Viète's formula:

$$S = \frac{a_1^2}{a_1 - a_2}$$

$$= \frac{a_1^2}{a_1 - r \cdot a_1}$$

$$= \frac{a_1^2}{a_1(1-r)}$$

$$S = \frac{a_1}{1-r}$$

Modern Formula

Power Series

Isaac Newton (1643 – 1727)

The Binomial Theorem:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \dots$$

Essentially, Newton arrived at this by observing a pattern in the expansions of binomials with positive integer exponents.

$$(1+x)^2 = 1 + 2x + x^2 = 1 + 2x + \frac{2\cdot 1}{2\cdot 1}x^2$$

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3 = 1 + 3x + \frac{3\cdot 2}{2\cdot 1}x^2 + \frac{3\cdot 2\cdot 1}{3\cdot 2\cdot 1}x^3$$

$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4 = 1 + 4x + \frac{4\cdot 3}{2\cdot 1}x^2 + \frac{4\cdot 3\cdot 2}{3\cdot 2\cdot 1}x^3 + \frac{4\cdot 3\cdot 2\cdot 1}{4\cdot 3\cdot 2\cdot 1}x^4$$

Power Series

Isaac Newton (1643 – 1727)

The Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \ldots$$

This can be applied to obtain lots of results.

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{(1/2)(-1/2)}{2}x^2 + \frac{(1/2)(-1/2)(-3/2)}{6}x^3 + \ldots$$

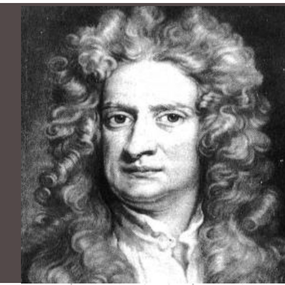
$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \ldots$$

$$(1-x^2)^{1/2} = 1 + \frac{1}{2}(-x^2) - \frac{1}{8}(-x^2)^2 + \frac{1}{16}(-x^2)^3 - \frac{5}{128}(-x^2)^4 + \ldots$$

$$= 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8 + \ldots$$

Power Series

Isaac Newton (1643 – 1727)



In search of a power series for acrsin(x)

$y = \arcsin(x) \qquad x = \sin(y)$

area = $\frac{y}{2} \cdot 1^2 = \frac{y}{2}$

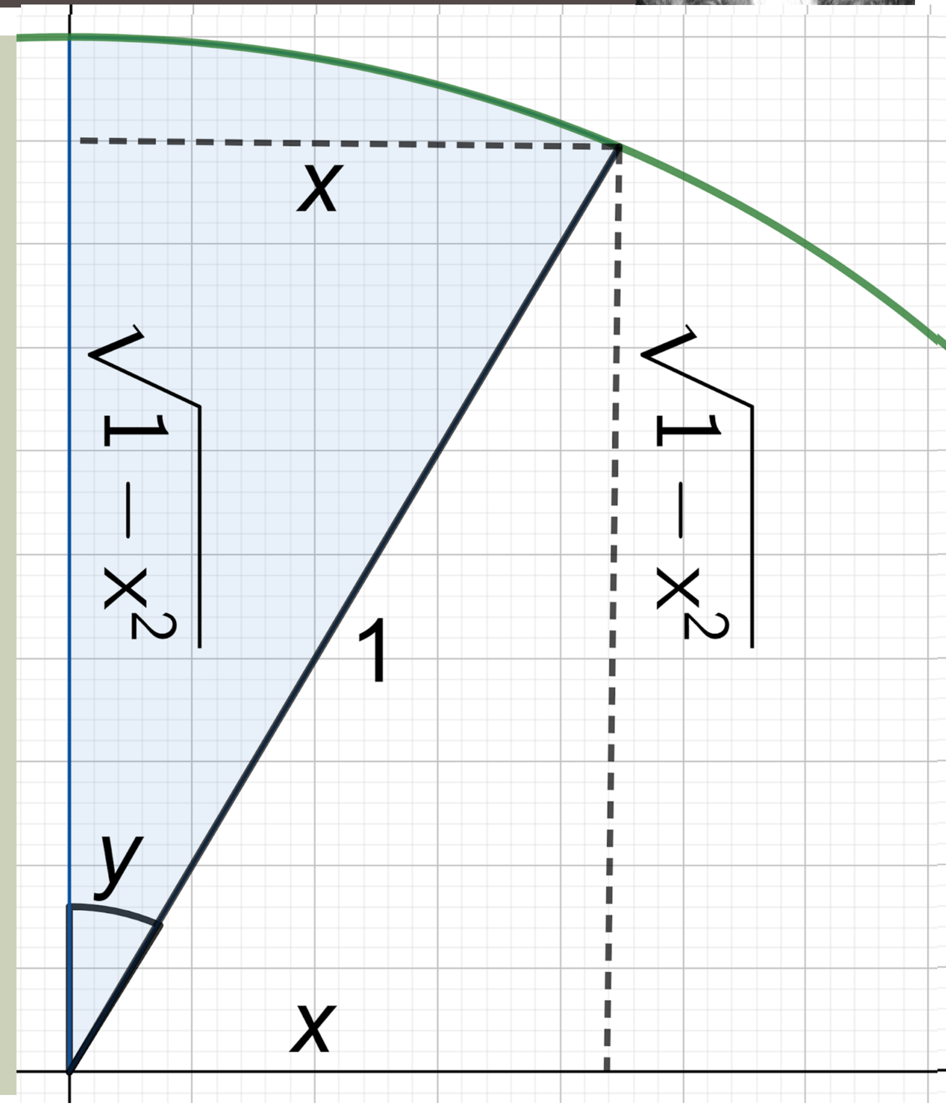
area = $\int_0^x \sqrt{1-t^2}\,dt$

$-\frac{1}{2} x \sqrt{1-x^2}$

$$\frac{y}{2} = \int_0^x \sqrt{1-t^2}\,dt - \frac{1}{2} x\sqrt{1-x^2}$$

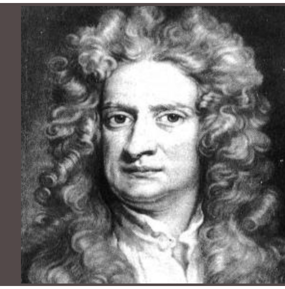
$$y = 2\int_0^x \sqrt{1-t^2}\,dt - x\sqrt{1-x^2}$$

arcsin(x)



Power Series

Isaac Newton (1643 – 1727)



In search of a power series for arcsin(*x*).

$$\arcsin(x) = 2\int_0^x \sqrt{1-t^2}\,dt - x\sqrt{1-x^2}$$

$$= 2\int_0^x \left(1 - \frac{1}{2}t^2 - \frac{1}{8}t^4 - \frac{1}{16}t^6 - \frac{5}{128}t^8 - \dots\right) dt$$

$$-x\left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8 - \dots\right)$$

$$= 2\left(x - \frac{1}{6}x^3 - \frac{1}{40}x^5 - \frac{1}{112}x^7 - \dots\right)$$

$$-\left(x - \frac{1}{2}x^3 - \frac{1}{8}x^5 - \frac{1}{16}x^7 - \dots\right)$$

$$= x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \dots$$

Power Series

Isaac Newton (1643 – 1727)

$$y = \arcsin(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \ldots$$

What about a power series for sin(*x*)?

Newton had a clever way to invert a power series.

It seems reasonable to assume that there should be a power series representation for *x* in terms of *y*.

$$x = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + \ldots$$

$$\sin(y) = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + \ldots$$

$$x = \sin(y)$$

$a_0 = 0$

Power Series

Isaac Newton (1643 – 1727)

In search of a power series for sin(*x*).

With $y = \arcsin(x)$ and $x = \sin(y)$,
Suppose we can find this power series $x = a_1 y + a_2 y^2 + a_3 y^3 + \ldots$

Remember $\arcsin(x) = y = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \ldots$

$$\begin{aligned} y = & \left(a_1 y + a_2 y^2 + a_3 y^3 + \ldots \right) \\ & + \tfrac{1}{6}\left(a_1 y + a_2 y^2 + a_3 y^3 + \ldots \right)^3 \\ & + \tfrac{3}{40}\left(a_1 y + a_2 y^2 + a_3 y^3 + \ldots \right)^5 \\ & + \tfrac{5}{112}\left(a_1 y + a_2 y^2 + a_3 y^3 + \ldots \right)^7 \\ & + \ldots \end{aligned}$$

Power Series

Isaac Newton (1643 – 1727)

In search of a power series for sin(*x*)

$$
\begin{aligned}
y = &\left(a_1 y + a_2 y^2 + a_3 y^3 + \dots\right) \\
&+ \tfrac{1}{6}\left(a_1 y + a_2 y^2 + a_3 y^3 + \dots\right)^3 \\
&+ \frac{3}{40}\left(a_1 y + a_2 y^2 + a_3 y^3 + \dots\right)^5 \\
&+ \frac{5}{112}\left(a_1 y + a_2 y^2 + a_3 y^3 + \dots\right)^7 \\
&+ \dots
\end{aligned}
$$

Expand this and collect the powers of *y*.

This gets messy quickly!

$$y = a_1 y + a_2 y^2 + \left(a_3 + \frac{a_1^3}{6}\right) y^3 + \left(\frac{a_2 a_1^2}{2} + a_4\right) y^4 + \dots$$

$a_1 = 1$

$a_2 = 0$

$a_3 + \frac{a_1^3}{6} = 0$

$a_3 = -\frac{1}{6}$

$\frac{a_2 a_1^2}{2} + a_4 = 0$

$a_4 = 0$

$a_5 = \frac{1}{120}$

Power Series

Isaac Newton (1643 – 1727)

In search of a power series for sin(*x*)

With $y = \arcsin(x)$ and $x = \sin(y)$,

Newton had finally arrived at his power series for $x = \sin(y)$.

$$x = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + \ldots$$

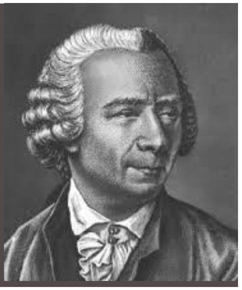
$a_0 = 0$, $a_1 = 1$, $a_2 = 0$, $a_3 = -\frac{1}{6}$, $a_4 = 0$, $a_5 = \frac{1}{120}$

$$\sin(y) = y - \frac{1}{6}y^3 + \frac{1}{120}y^5 \ldots$$

$$= y - \frac{1}{3!}y^3 + \frac{1}{5!}y^5 - \frac{1}{7!}y^7 + \ldots$$

The Basel Problem

Leonhard Euler (1707-1783)



$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots$$

This sum was thought to resolve to a value around 1.625.

But what *was* this value?

Euler’s fantastic approach to this problem begin with exploring Newton’s power series for sin(x).

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots$$

$$\frac{\sin(x)}{x} = 1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{1}{7!}x^6 + \ldots$$

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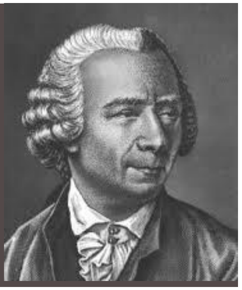


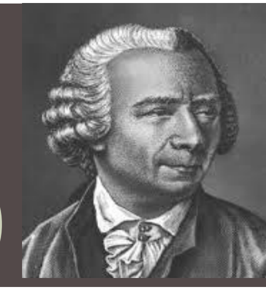
Factor Theorem [Rene Descartes (1596 - 1650)]

The polynomial *f*(*x*) has a factor of (*x* - *k*) if and only if *f*(*k*) = 0 (that, is *k* is a zero/root of the polynomial).

Equivalently, the polynomial *f*(*x*) has a factor of $\left(1 - \frac{x}{k}\right)$ if and only if *f*(*k*) = 0.

The Basel Problem

Leonhard Euler (1707-1783)

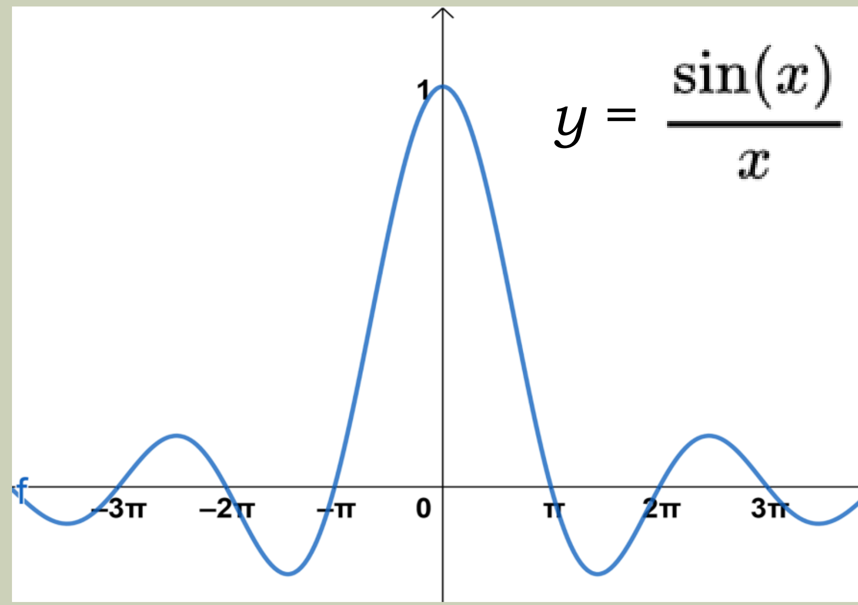


The polynomial *f*(*x*) has a factor of $\left(1-\frac{x}{k}\right)$ **if and only if *f*(*k*) = 0.**

Does this extend in an infinite way?

Euler: Why not?

Us: Be careful…

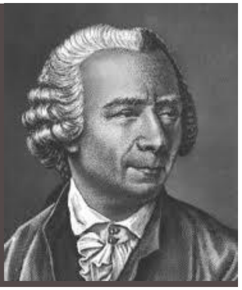


$a = 1$

$$\frac{\sin(x)}{x} = a\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\cdots$$

The Basel Problem

Leonhard Euler (1707-1783)



$$\frac{\sin(x)}{x} = \left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\dots$$

$$= \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\left(1-\frac{x^2}{16\pi^2}\right)\dots$$

$$= \left(1-\frac{x^2}{\pi^2}-\frac{x^2}{4\pi^2}+\frac{x^4}{4\pi^4}\right)\left(1-\frac{x^2}{9\pi^2}\right)\left(1-\frac{x^2}{16\pi^2}\right)\dots$$

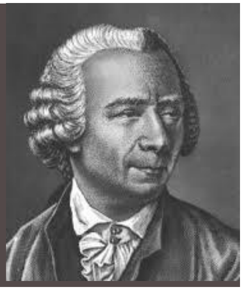
$$= \left(1-\frac{x^2}{\pi^2}-\frac{x^2}{4\pi^2}-\frac{x^2}{9\pi^2}+f(x^4)+g(x^6)\right)\left(1-\frac{x^2}{16\pi^2}\right)\dots$$

$$\vdots$$

$$\frac{\sin(x)}{x} = 1 - x^2\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\frac{1}{16\pi^2}+\dots\right)+a_1x^4+a_2x^6+\dots$$

The Basel Problem

Leonhard Euler (1707-1783)



$$\frac{\sin(x)}{x} = 1 - x^2\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \ldots\right) + a_1 x^4 + a_2 x^6 + \ldots$$

But Euler already had a power series for $\frac{\sin(x)}{x}$.

$$\frac{\sin(x)}{x} = 1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{1}{7!}x^6 + \ldots$$

We must be obtaining the same power series, so the coefficients must be the same on each power of *x*.

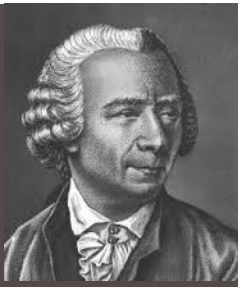
$$\frac{1}{3!} = \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \ldots$$

$$\frac{1}{6} = \frac{1}{\pi^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots\right)$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$$

The Basel Problem

Leonhard Euler (1707-1783)



$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots = \frac{\pi^2}{6}$$

Euler was able to mimic this approach to give the result for any even exponent.

Finding an exact value for $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is still an open problem.

In fact, exact values for this sum are not known for any odd exponents.

Questions or Comments?

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Sources

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