All slides in this talk have to read up-side down ☹

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7e3

!Unexpected Surprises!

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$\sqrt{4076361}$

Outline

1. Extended Japanese Sangaku Theorem from 1800

2. 1972 R.P. Stanley’s theorem on partitions

3. Average number of ways integers 0 thru (n – 1) can be expressed as the sum of two (ordered) integral squares (Gauss ~1800)

Sources:

“Mathematical Gems III”, “Ingenuity in Mathematics” by Ross Honsberger

“Advanced Euclidean Geometry” by R.A. Johnson (1961, orig. 1929)

You can present this to your students… Beautiful high school math here ☺

Please feel free to ask questions at any time ☺

1. 1800 Japanese Theorem

Take any n sided convex polygon inscribed in a circle and triangulate it using any one vertex. Inscribe circles in each of the (n – 2) triangles.

The sum of the radii of the inscribed circles is a constant – regardless of which vertex is chosen!

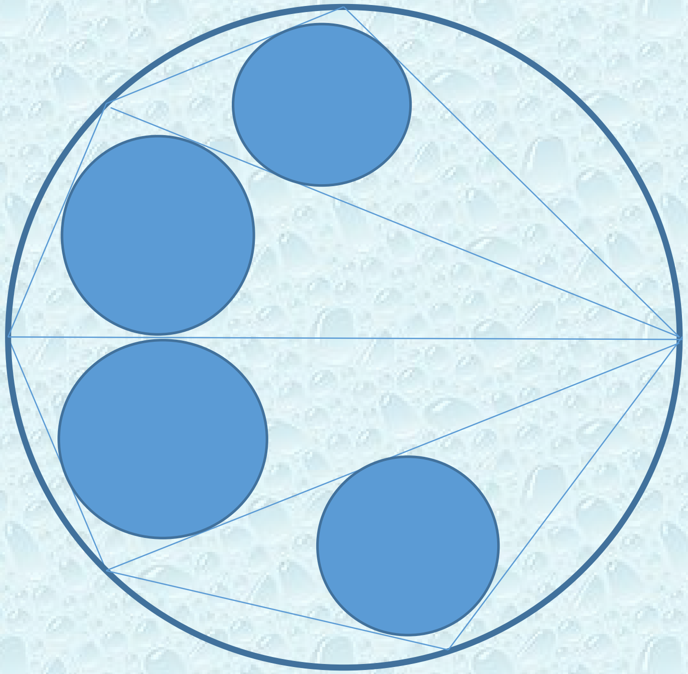
1800 Japanese Theorem (extended)

Take any n sided convex polygon inscribed in a circle and triangulate it using one or more vertices. Inscribe circles in each of the (n – 2) triangles.

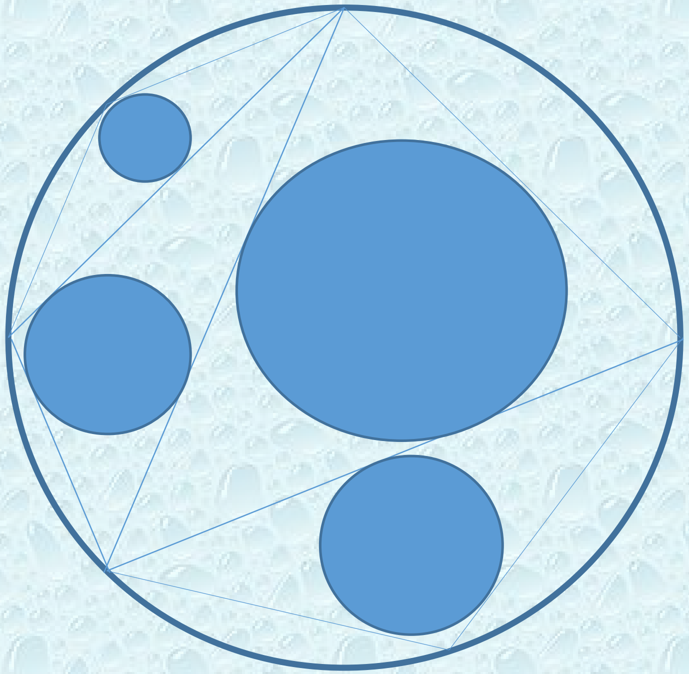
The sum of the radii of the inscribed circles is a constant --- regardless of which triangulation is chosen!

Example:
6 sided polygon results in 4 triangles

Original Triangulation Method



"Extended" Triangulation



Sum of the radii is a <u>constant</u> <u>regardless</u> of the Triangulation

Sangaku Mathematics (Wikipedia)

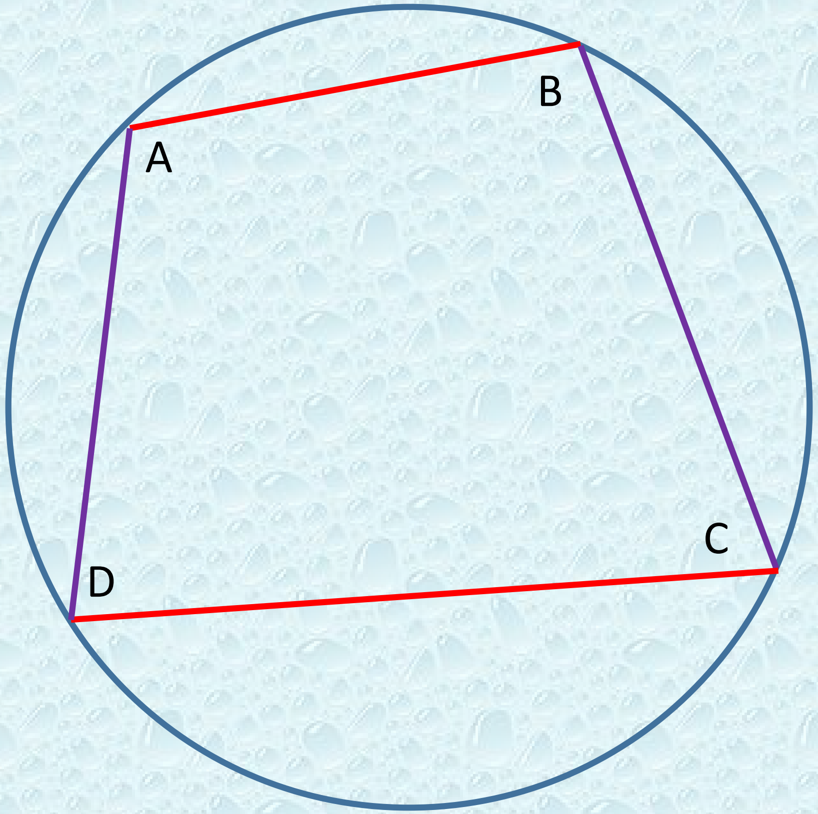
Sangaku or **San Gaku** (算額; lit. translation: calculation tablet) are Japanese geometrical problems or theorems on wooden tablets which were placed as offerings at Shinto shrines or Buddhist temples during the Edo period (1603 – 1868) by members of all social classes.

- The Sangaku were painted in color on wooden tablets (ema) and hung in the precincts of Buddhist temples and Shinto shrines as offerings to the kami and buddhas, as challenges to the congregants...
- Fujita Kagen (1765–1821), a Japanese mathematician of prominence, published the first collection of *sangaku* problems, his *Shimpeki Sampo* (Mathematical problems Suspended from the Temple) in 1790, and in 1806 a sequel, the *Zoku Shimpeki Sampo*.
- During this period Japan applied strict regulations to commerce and foreign relations for western countries so the tablets were created using Japanese mathematics, developed in parallel to western mathematics. For example, the connection between an integral and its derivative (the fundamental theorem of calculus) was unknown, so Sangaku problems on areas and volumes were solved by expansions in infinite series and term-by-term calculation.

First some geometry background.

This should bring back some pleasant memories for some of us. ☺

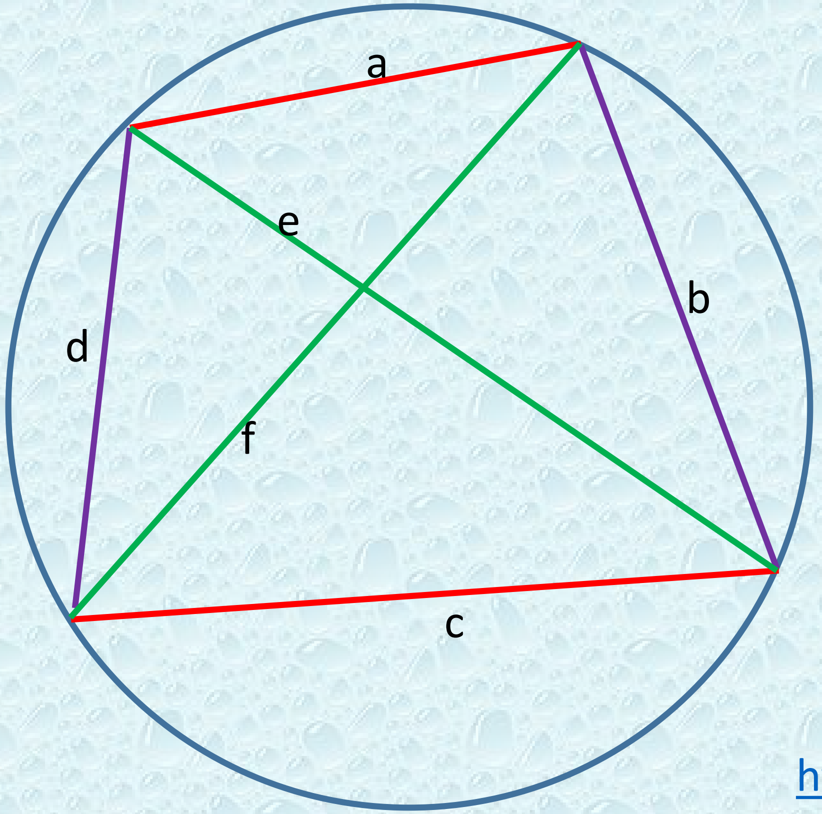
1. An <u>iff</u> property of cyclic quadrilaterals



$$A + C = B + D = 180^{\circ}$$

https://polymathematics.typepad.com/polymath/cyclic-quadrilaterals.html

2. Ptolemy's theorem: Cyclic quadrilaterals



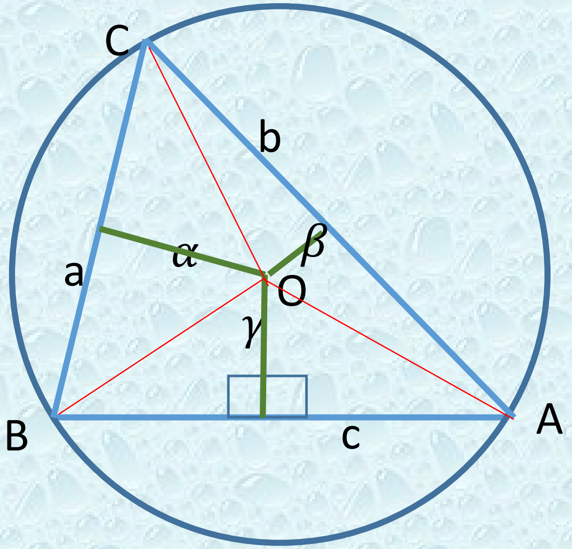
$$ac + bd = ef$$

https://www.cut-the-knot.org/proofs/ptolemy.shtml

3. Area of the triangle

Perpendicular bisectors

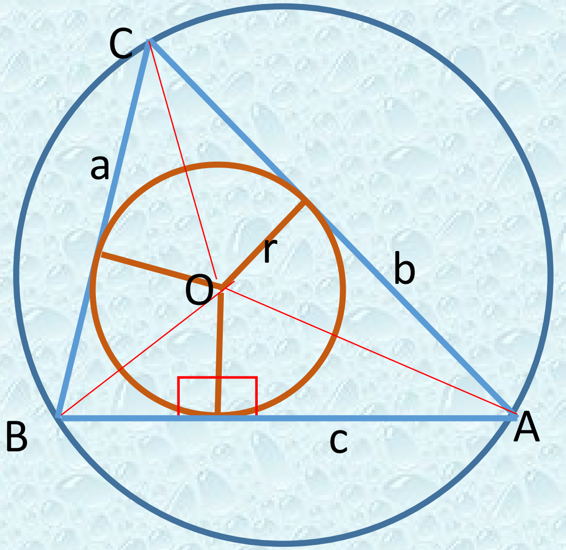
Circumcenter



$$\text{Area} = \alpha\frac{a}{2} + \beta\frac{b}{2} + \gamma\frac{c}{2}$$

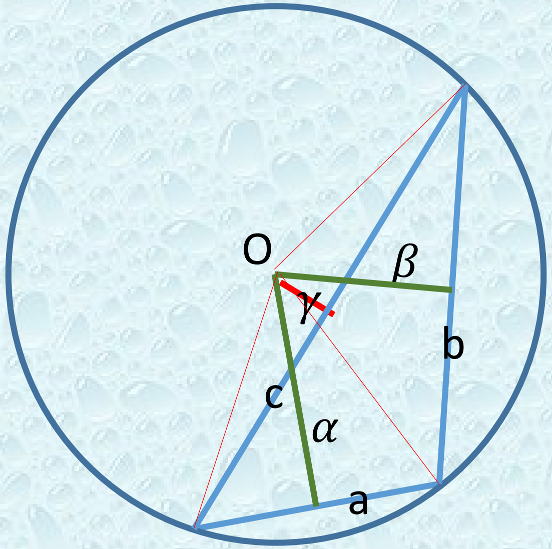
Angular bisectors

Incenter



$$\text{Area} = r\,\frac{a+b+c}{2}$$

4. Area of the triangle
Perpendicular bisector outside the triangle



$$\text{Area} = \alpha\,\frac{a}{2} + \beta\,\frac{b}{2} - \gamma\,\frac{c}{2}$$

$$= r\,\frac{a+b+c}{2} \quad \text{(unchanged)}$$

We also need the beautiful theorem by Carnot before we get to our extended Japanese Theorem.

5. L.N.M Carnot's (1753-1823) Theorem

Statement:

In any triangle ABC, the sum of the signed distances from the circumcenter O to the sides is R + r

R = circumradius

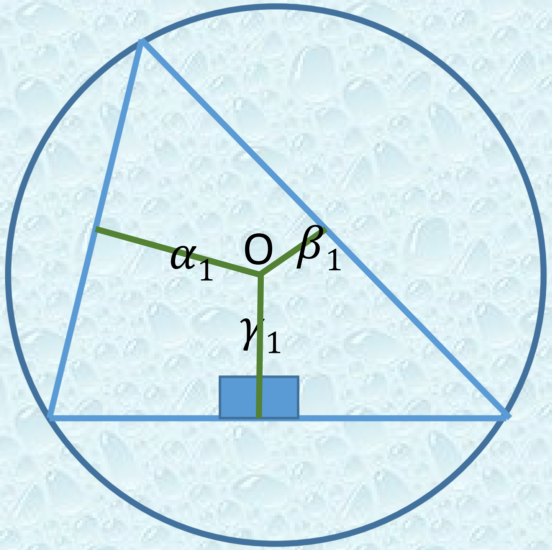
r = inradius

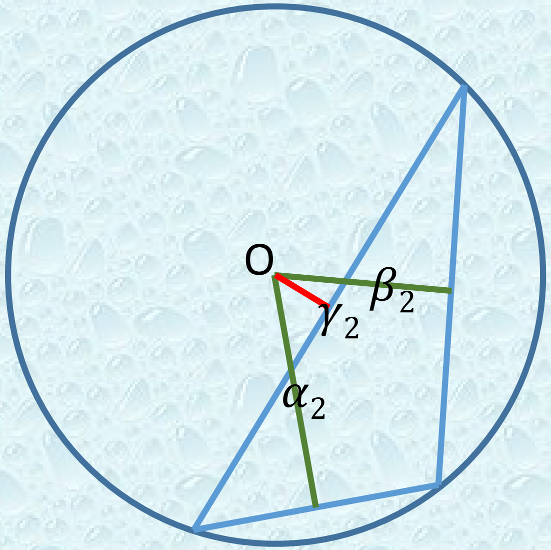
https://demonstrations.wolfram.com/CarnotsTheorem/

5. Carnot's theorem: Signed sum of perpendicular bisectors

Positive distance

Negative distance
(if completely outside the triangle)

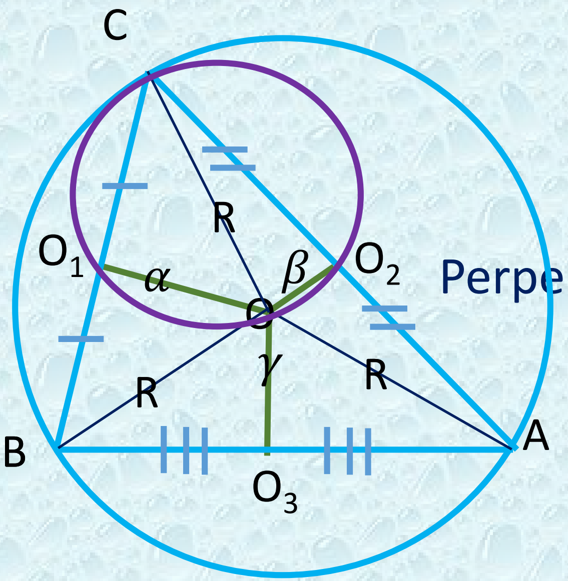


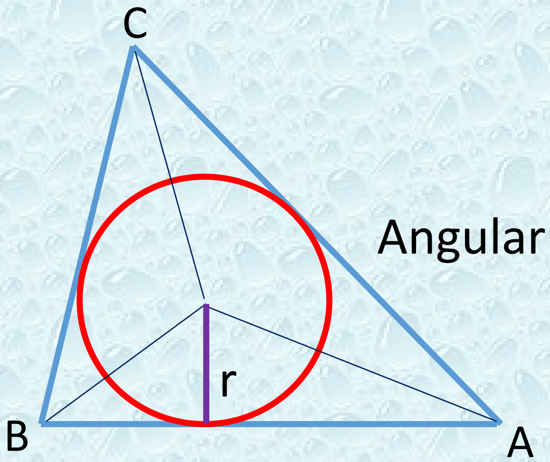


$$\alpha_1 + \beta_1 + \gamma_1 = \mathrm{R} + \mathrm{r}_1$$

$$\alpha_2 + \beta_2 - \gamma_2 = \mathrm{R} + \mathrm{r}_2$$

Beautiful Proof of Carnot's theorem: Positive Distances





cyclic quad: O_1CO_2O: $R\frac{c}{2} = \beta\frac{a}{2} + \alpha\frac{b}{2}$

O_2AO_3O: $R\frac{a}{2} = \gamma\frac{b}{2} + \beta\frac{c}{2}$

O_3BO_1O: $R\frac{b}{2} = \alpha\frac{c}{2} + \gamma\frac{a}{2}$

Adding: $R\frac{a+b+c}{2} = \alpha(\frac{b}{2}+\frac{c}{2}) + \beta(\frac{c}{2}+\frac{a}{2}) + \gamma(\frac{a}{2}+\frac{b}{2})$ (1)

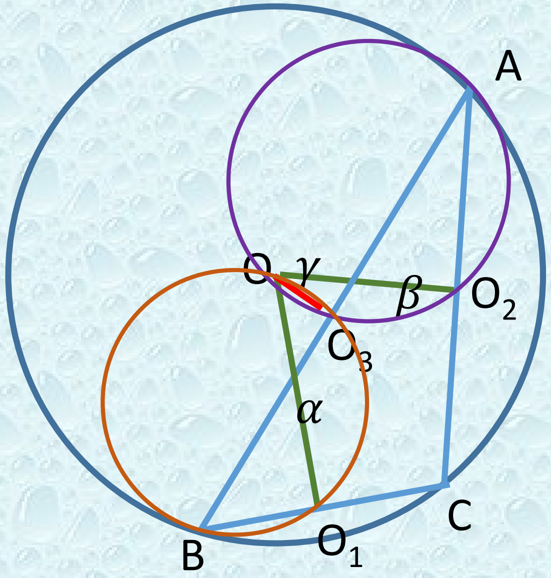
area of triangle = $r\frac{a+b+c}{2} = \alpha\frac{a}{2} + \beta\frac{b}{2} + \gamma\frac{c}{2}$ (2)

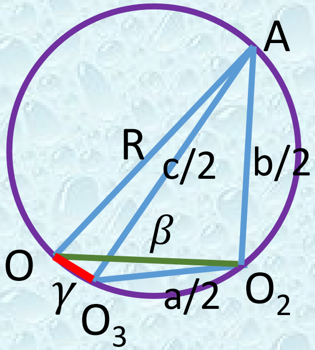
Adding (1) and (2)

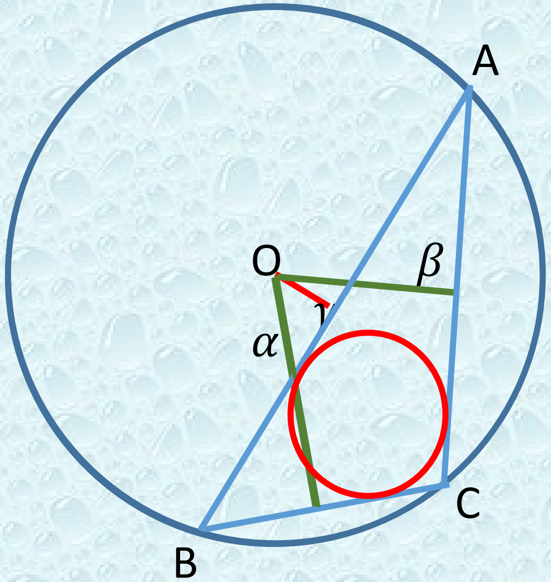
$(R+r)\frac{a+b+c}{2} = (\alpha+\beta+\gamma)\frac{a+b+c}{2}$

$\boxed{R + r = \alpha + \beta + \gamma}$ ☺

Carnot's theorem (Negative distance)







cyclic quad: O_1CO_2O: $R\,\frac{c}{2} = \beta\,\frac{a}{2} + \alpha\,\frac{b}{2}$

(in semi circle) O_2AOO_3: $R\,\frac{a}{2} = -\gamma\,\frac{b}{2} + \beta\,\frac{c}{2}$

(in semi circle) O_3OBO_1: $R\,\frac{b}{2} = \alpha\,\frac{c}{2} - \gamma\,\frac{a}{2}$

Adding: $R\,\frac{a+b+c}{2} = \alpha(\frac{b}{2} + \frac{c}{2}) + \beta(\frac{c}{2} + \frac{a}{2}) - \gamma(\frac{a}{2} + \frac{b}{2})$ (1)

$r\,\frac{a+b+c}{2} = \alpha\,\frac{a}{2} + \beta\,\frac{b}{2} - \gamma\,\frac{c}{2}$ (2)

Adding (1) and (2)

$(R+r)\,\frac{a+b+c}{2} = (\alpha + \beta - \gamma)\,\frac{a+b+c}{2}$

$\boxed{R + r = \alpha+\beta-\gamma}$ ☺

Back to our Japanese theorem (finally!)
Proof that the sum of the radii is a constant

Every triangle in any triangulation has the same circumcircle (the outer circle) with radius R

Let the radii of the inscribed circles be r_1, r_2, r_3, …

Let the signed distances from the center of the circumcircle to sides of the inscribed triangle i be α_i, β_i, and γ_i

$$R + r_i = \alpha_i + \beta_i \pm \gamma_i \text{ for each of the inscribed triangles}$$

Proof: Continued

$R + r_i = \alpha_i + \beta_i + \gamma_i$ for each of the $(n - 2)$ inscribed triangles

$r_i = \alpha_i + \beta_i + \gamma_i - R$

$S = \sum r_i = \sum (\alpha_i + \beta_i + \gamma_i) - (n - 2)R$

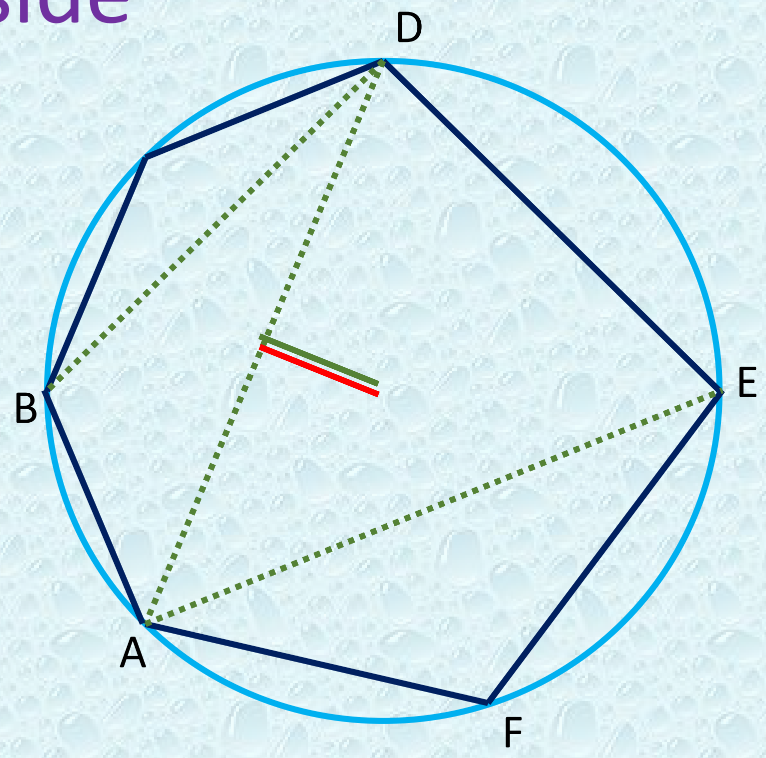
Need to show that $S' = \sum (\alpha_i + \beta_i + \gamma_i)$ is a constant regardless of the triangulation.

Perpendicular bisector on a diagonal of the triangulation

The bisector on the diagonal AD is inside Δ ADE and hence +ve.

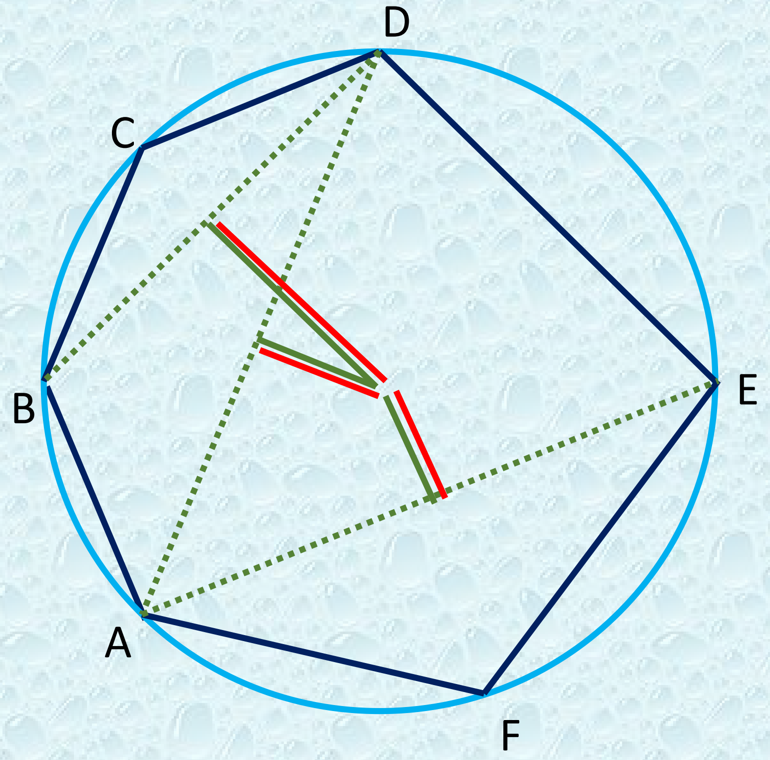
The same bisector on AD is outside Δ ABD and hence –ve.

So these will cancel each other out.



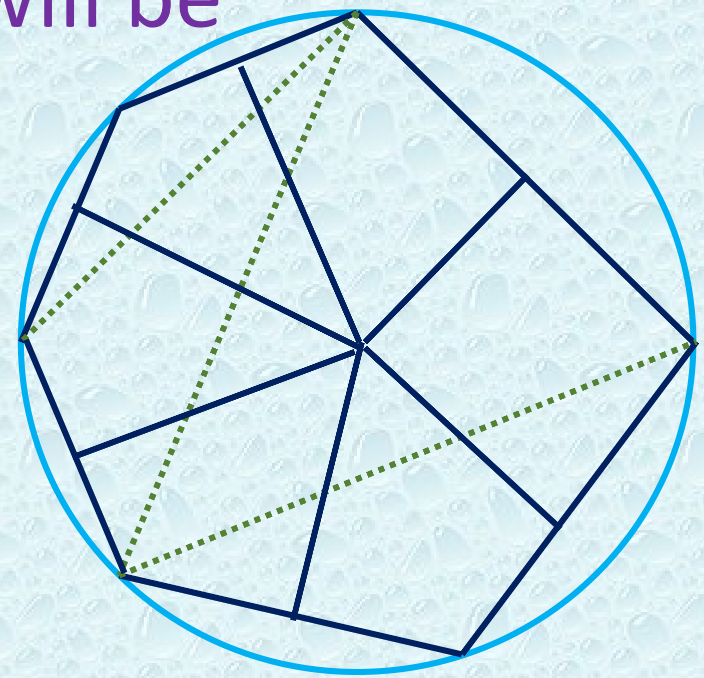
Perpendicular bisectors on the diagonals of the triangulation

The bisectors on the diagonals occur in pairs and will pairwise cancel each other.



The perpendicular bisectors on the edges of the polygon

The bisectors on the edges of the polygon (always +ve distances) occur once and will be preserved in the sum.

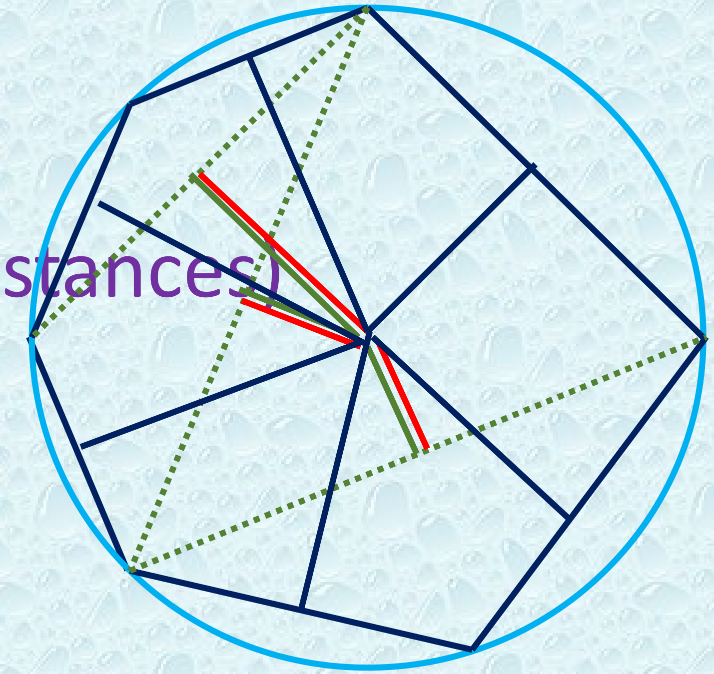


Triangulations do not affect the bisectors to the edges of the polygon.

Add all the perpendicular bisectors

The bisectors of the diagonals of the triangulation occur in pairs and cancel each other.

The bisectors to the outer edges (positive distances) occur once and are preserved by the sum.



Hence the sum will be a constant. QED!!

1972 R.P. Stanley’s theorem on partitions of integers

Partitions are unordered parts of an integer that add up to the number

1 = 1 p(1) = 1

2 = 1 + 1 = 2 p(2) = 2

3 = 1 + 1 + 1 = 2 + 1 = 3 p(3) = 3

4 = 1 + 1 + 1 + 1 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 4 p(4) = 5

This number grows pretty rapidly and is denoted by p(n)

p(n) values for n = 0, 1, 2, …

1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, 1958, 2436, 3010, 3718, 4565, 5604, …
(sequence A000041 in the OEIS)

Rich field of study in Number Theory – S. Ramanujan

2. 1972 R.P. Stanley's theorem

The total number of 1's among all unordered partitions of a positive integer is equal to the sum of the numbers of distinct parts of those partitions.

A theorem discovered after I was born and one that I can understand ☺

Unlikely there are more than a handful of these, if any ☹

Example of Stanley's Thm. (note the order!)

p (5) = 7

	distinct items	# of distinct items
5 = 1 + 1 + 1 + 1 + 1	1	1
= 1 + 1 + 1 + 2	1, 2	2
= 1 + 1 + 3	1, 3	2
= 1 + 2 + 2	1, 2	2
= 1 + 4	1, 4	2
= 2 + 3	2, 3	2
= 5	5	1

Number of 1's = 12

Total number of distincts = 12

History from Wolfram Math

The general result was discovered by R. P. Stanley in 1972 and submitted to the "Problems and Solutions" section of the American Mathematical Monthly, where it was rejected with the comment "A bit on the easy side, and using only a standard argument," presumably because the editors did not understand the actual statement and solution of the problem

Will motivate the proof using our p(5) example

Proof: Arrange the partitions in a non decreasing order in a table

Partition #	col 1	col 2	col 3	col 4	col 5
1.	1	1	1	1	1
2.	1	1	1	2	
3.	1	1	3		
4.	1	2	2		
5.	1	4			
6.	2	3			
7.	5				
# of 1's/col.	5	3	2	1	1

Let’s focus on the columns of this table.

We will count how many 1’s are in each column.

# of 1's in column 3 = p (5 – 3) = p(2)

Partition #	col 1	col 2	Focus on col. k = 3	col 3	col 4	p (5 - 3) = p (2) = 2
1.	1	1	1	1	1	1 + 1 = 2
2.	1	1	1	2		2 = 2
3.	1	1	3			
4.	1	2	2			
5.	1	4				
6.	2	3				
7.	5					
# of 1's/col	5	3	2	1	1	

Numbers in each red box add up to k.

Note: # of 1's in column 5 = 1 = p (5 – 5) = p (0)

of 1’s in column 2 = 3 = p (5 – 2) = p(3)

Partition #	col 1	Focus on col. k = 2	col 3	col 4	col 5	p (5 - 2) = p (3) = 3
1.	1	1	1	1	1	1 + 1 + 1 = 3
2.	1	1	1	2		1 + 2 = 3
3.	1	1	3			3 = 3
4.	1	2	2			
5.	1	4				
6.	2	3				
7.	5					
# of 1’s/col.	5	3	2	1	1	

Generalizing, we see that number of 1’s in column k = p(n – k)

Adding up the 1’s in all the columns

From the previous slides, number of 1’s in column k = p(n – k)

Total number of 1’s in all columns = $\sum_{k=1}^{n} p(n-k)$ (1)

Next, we count the number of distinct parts in each partition.

Number of distinct parts in each partition:

Let's focus on some part, say 2.

(Ignore duplicates in the same row)

Partition #	col 1	col 2	col 3	col 4	col 5	**p (5 - 2) = p (3) = 3**
1.	1	1	1	1	1	
2.	1	1	1	2		1 + 1 + 1 = 3
3.	1	1	3			
4.	1	2	2			1 + 2 = 3
5.	1	4				
6.	2	3				3 = 3
7.	5					
# of 1's/col.	5	3	2	1	1	

Thus 2 occurs as a distinct part in p (5 – 2) rows

Similarly,
number of rows containing 1 = p (5 – 1) = p(4)

Partition #	col 1	col 2	col 3	col 4	col 5	**p (5 - 1) = p (4) = 5**
1.	1	1	1	1	1	1 + 1 + 1 + 1 = 4
2.	1	1	1	2		1 + 1 + 2 = 4
3.	1	1	3			1 + 3 = 4
4.	1	2	2			2 + 2 = 4
5.	1	4				4 = 4
6.	2	3				
7.	5					
# of 1's/col.	5	3	2	1	1	

Thus 1 occurs as a distinct part in p (5 – 1) rows

Conclusion!

Number of rows containing k = p (n – k)

Hence number of distinct elements in all rows = $\sum_{k=1}^{n} p(n-k)$

= Total number of 1's in all columns ☺

Observation: Proof is mesmerizingly simple! (Once you see it)

A comment in the "Math Gems III" reads: "This argument was derived from independent proofs by two outstanding mathematicians E.W. Dijkstra and Prof. K.A. Post".

Why did this need such "heavy hitters"?

A Partition Problem for your students!

The number of partitions of an integer n in which all parts are odd
equals
the number of partitions of n in which all parts are distinct.

$5 = 1 + 1 + 1 + 1 + 1$	all odd	
$= 1 + 1 + 1 + 2$		
$= 1 + 1 + 3$	all odd	
$= 1 + 2 + 2$		
$= 1 + 4$		all parts distinct
$= 2 + 3$		all parts distinct
$= 5$	all odd	all parts distinct

3. Sum of two squares

Average number of ways integers 0 thru (n – 1) can be expressed as the sum of two integral squares = ?

Any guesses?

If you already know the answer, silence please ☺

We start with the function r(n)

r (n) = number of ways any non negative integer n can be expressed as the (ordered) sum of two integral squares

r (5) = 8

Ordered Pairs:

$5 = (+1)^2 + (+2)^2 = (+2)^2 + (+1)^2$	(+1,+2), (+2, +1)	Q1,Q1
$= (-1)^2 + (+2)^2 = (+2)^2 + (-1)^2$	(-1,+2), (+2, -1)	Q2,Q4
$= (+1)^2 + (-2)^2 = (-2)^2 + (+1)^2$	(+1,-2), (-2, +1)	Q4,Q2
$= (-1)^2 + (-2)^2 = (-2)^2 + (-1)^2$	(-1,-2), (-2, -1)	Q3,Q3

2 pairs in each quadrant

Behavior of r(n) is erratic!

r (0) = 1, r(1) = 4, r(2) = 4, r(3) = 0, r(4) = 4; r(5) = 8, r(7) = 0, r(12) = 0

Clearly r(4n+3) = 0; 3, 7, 11 …

r((4n+3)2^m) = 0, m ≥ 0; Why??? 6, 14, 22, …

How high can r(n) go? The sky is the limit!

Why is there no upper bound?
$r(p = 4n+1) = 8$ (from Number Theory)

$r(pq) = 16$; $r(pqs) = 32$ and so on (q, s … $4n+1$ primes)

$5 = 1^2 + 2^2$; $13 = 2^2 + 3^2$; $17 = 4^2 + 1^2$

$65 = 5 \cdot 13 = 4^2 + 7^2 = 1^2 + 8^2$

$1105 = 5 \cdot 13 \cdot 17 = 4^2 + 33^2 = 9^2 + 32^2 = 12^2 + 31^2 = 23^2 + 24^2$

Let p, q = different $4n+1$ primes; $p = a^2 + b^2$ and $q = c^2 + d^2$

$p = (a + ib)(a – ib)$ and $q = (c + id)(c – id)$

$pq = (a + ib)(c+id)\,(a – ib)(c – id) = (a + ib)(c – id)\,(a – ib)(c + id)$

!Conjugates!

Since r(n) is highly irregular, we look at its average

The average is surprisingly well behaved!

Consider $\dfrac{r(0) + r(1) + r(2) + \ldots + r(z-1)}{z} = \dfrac{R(z)}{z}$

We now study $\lim\limits_{z \to \infty} \dfrac{R(z)}{z}$. This limit exists!!

And the proof (Gauss, ~1800, 23 years old) is easy to follow ☺

Surprised that it took someone of Gauss's caliber to solve this…

Proof

Consider the circle $C(\sqrt{z})$: $x^2 + y^2 = z$

A lattice point (a, b) has integer coordinates a and b

Every lattice point <u>inside</u> the circle satisfies $a^2 + b^2 < z$

Also $a^2 + b^2 = n < z$ is an integer

Every lattice point inside the circle contributes 1 to $R(z)$ as it is a pair counted by some $r(n)$, $n < z$.

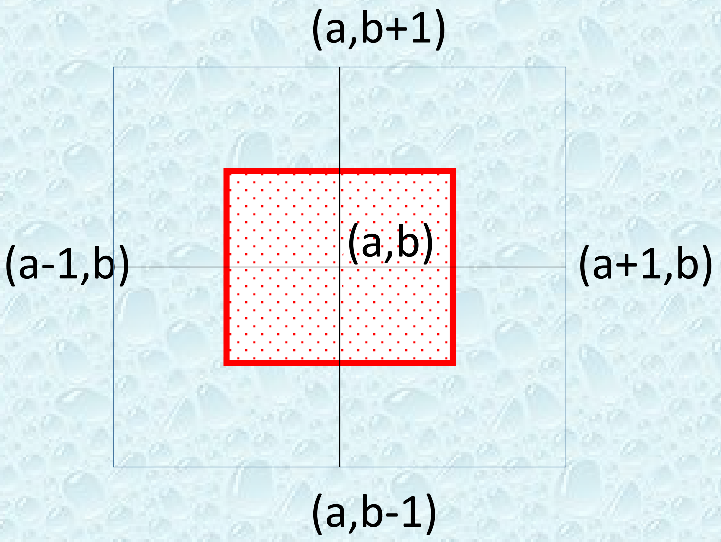
Conversely, any point counted by $R(z)$ is a lattice point

Number of lattice points <u>inside</u> $C(\sqrt{z}) = R(z)$

How many lattice points are inside $C(\sqrt{z})$?

Consider any lattice point (a, b) inside the circle

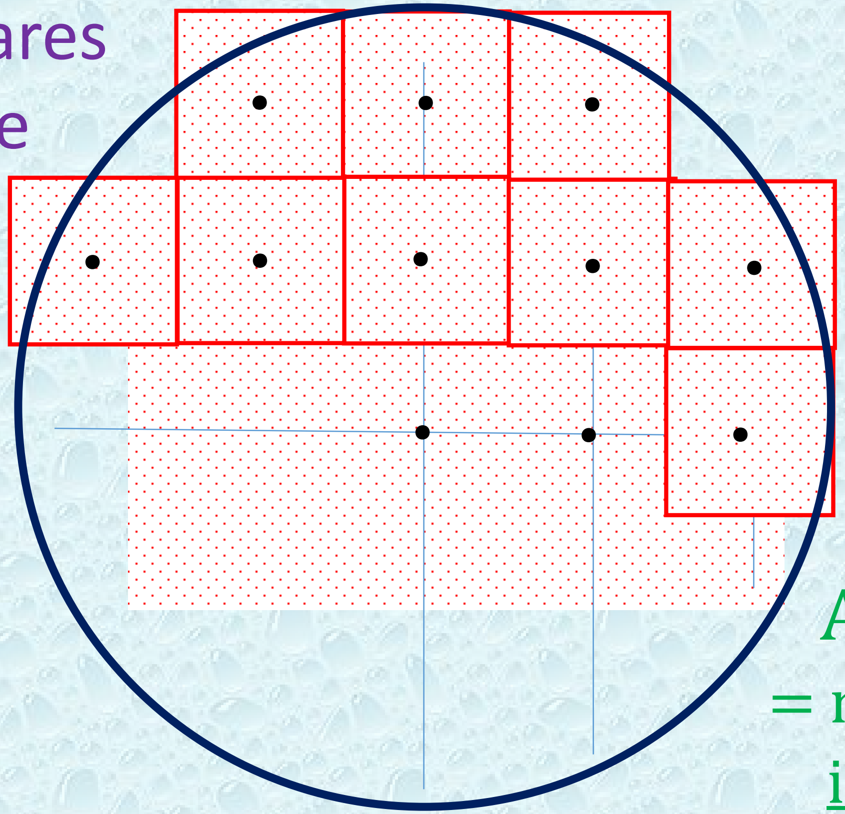
Draw a red square with a side length of 1 (a unit square)



Draw unit red squares around all lattice points inside the circle

Some of the red squares are sticking out of the circle.

Some parts of the circle are empty.



Area of red region

= number of lattice points <u>inside</u> the circle

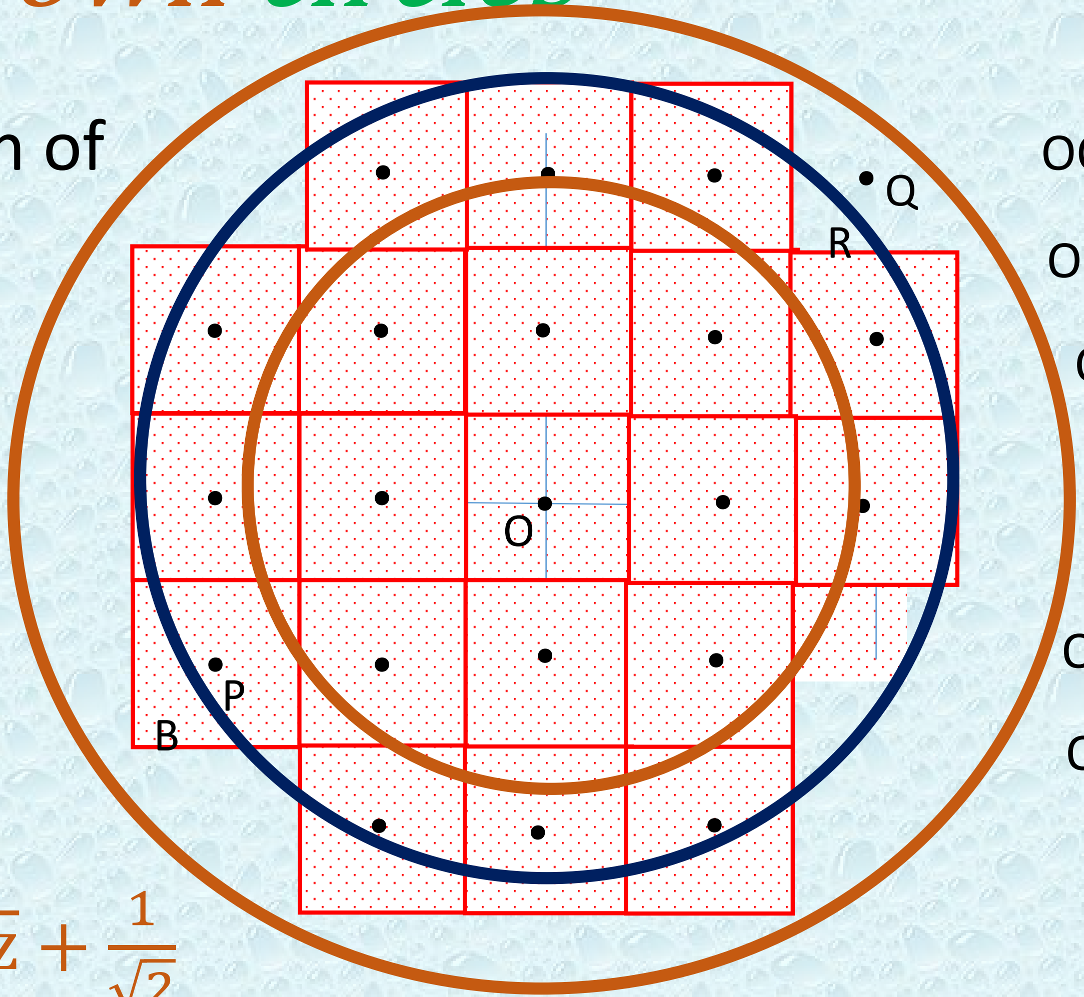
= R(z)!!

Area of red region lies between the areas of the two brown circles

½ diagonal length of unit square = $\frac{1}{\sqrt{2}}$

Radius of inner brown circle $= \sqrt{z} - \frac{1}{\sqrt{2}}$

Radius of outer brown circle $= \sqrt{z} + \frac{1}{\sqrt{2}}$



$OQ \geq \sqrt{z}$; $RQ \leq \frac{1}{\sqrt{2}}$

$OR + RQ \geq OQ$ (Δ inequality)

$OR \geq OQ - RQ \geq \sqrt{z} - \frac{1}{\sqrt{2}}$

$OP < \sqrt{z}$; $PB \leq \frac{1}{\sqrt{2}}$

$OB \leq OP + PB < \sqrt{z} + \frac{1}{\sqrt{2}}$

As z gets larger… Area of red region gets closer to the area of the circle $C(\sqrt{z}) = \pi z$

Area of $C\left(\sqrt{z} - \frac{1}{\sqrt{2}}\right) \leq$ Area of Red Region = $R(z) \leq$ Area of $C\left(\sqrt{z} + \frac{1}{\sqrt{2}}\right)$

$$\pi\left(z + \frac{1}{2} - \sqrt{2z}\right) \leq R(z) \leq \pi\left(z + \frac{1}{2} + \sqrt{2z}\right)$$

$$\pi\left(1 + \frac{1}{2z} - \sqrt{\frac{2}{z}}\right) \leq \frac{R(z)}{z} \leq \pi\left(1 + \frac{1}{2z} + \sqrt{\frac{2}{z}}\right)$$ (Squeeze theorem!)

$$\lim_{z \to \infty} \frac{R(z)}{z} = \pi$$ ☺

How fast do we get to π?

z	R(z)/z	Unordered in Q1 only
10	2.9	0.7
10^2	3.05	0.47
10^3	3.133	0.419
10^4	3.1397	0.4100
10^5	3.14173	0.39542
10^6	3.141521	0.393544
10^7	3.1415993	0.3929699
10^8	3.14159017	0.39278413
10^9	3.141592369	0.392726038 (*8 = 3.1418)

Cost of gas at Costco the other day = $3.149 ☺

A simple surprise for you!
Add the numbers of each color

29				
13	27			
11	15	25		
3	9	17	23	
1	5	7	19	21

Very “Cubicle”!!

	29					
	13	27				
	11	15	25			
	3	9	17	23		
	1	5	7	19	21	
Total	1	8	27	64	125	☺

???s

Thank you ☺